Principles of Algebra 2

APPLIED ALGEBRA FROM A BIBLICAL WORLDVIEW

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Some review information (including various definitions) was adapted/expanded from *Principles of Mathematics* (Katherine A. Loop, *Principles of Mathematics*, Master Books, Green Forest, AR: 2015-2016); please see *Principles of Mathematics* for more details on math's foundations (on which algebra builds).

Note: In putting this material together, many different resources were consulted, many of which are footnoted where appropriate. We do not necessarily recommend these materials; while they were consulted for facts, some do not claim to be from a biblical worldview and should be approached with discernment.

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Throughout the book, different mathematical concepts are referred to as “tools.” This analogy stems from Walter W. Sawyer’s profound comparison:

> *“Mathematics is like a chest of tools: Before studying the tools in detail, a good workman should know the object of each, when it is used, how it is used, what it is used for.”*[1]

It’s our prayer that this book will equip you to use algebra effectively.

About the Authors

Katherine [Loop] Hannon has been writing and speaking about math for more than 15 years. Understanding the biblical worldview in math during her senior year of homeschool made a tremendous difference in her life and started her on a journey of researching and sharing on the topic. Her books on math and a biblical worldview have been used by various Christian colleges, homeschool groups, and individuals.

Dr. Adam Floyd Hannon, Katherine’s husband, has long had a passion for math and science. He obtained his doctor of science (ScD) degree in materials science and engineering from the Massachusetts Institute of Technology after obtaining bachelor of science (BS) degrees in both physics and polymer and fiber engineering from the Georgia Institute of Technology. He currently works as a data scientist, applying many mathematical tools and algorithms to help find fraud in the healthcare system.

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[1] Walter W. Sawyer, *Mathematician’s Delight* (Harmondsworth Middlesex: Penguin, 1943), p. 10, quoted in James D. Nickel, rev. ed., *Mathematics: Is God Silent?* (Vallecito, CA: Ross House Books, 2001), p. 290.

Preface

Cokie Roberts once said that "as long as there is algebra in school, there will be prayer in school." While I believe Cokie meant that as a statement about how algebra would make students pray for help in understanding it, it's our hope that this curriculum will help students pray during their study of algebra out of wonder and awe at God's handiwork instead.

Rather than simply presenting apparently meaningless facts and problems to solve, our aim in this program is to take students on a journey into discovering how math helps us describe God's creation and gives us a glimpse into just how faithful and incredible He is. To do this, we'll be synthesizing information that most students never hear about unless they pursue a degree in a technical field.

It's our earnest prayer that you will be blessed by this project.

Soli Deo Gloria,
Katherine and Dr. Adam Hannon

Acknowledgments

Many people have contributed their time, prayers, and God-given talents to make this work a reality. We would like to specifically acknowledge several people who played a key role in its development:

- Jamin Pratt and Michael Ferreira, for testing the initial draft of the material for us, providing valuable insights from a student's perspective to help us adjust and refine. Sue Desmarais, a seasoned math teacher at both the high school and college level, for her valuable edits and polishing of the final product; Joy Dubbs, an engineer and friend, for hers; Hank Evans, Adam's high school Algebra 2 teacher, for his helpful thoughts on the overall outline; Brian Loop, Kate's engineering brother, for his availability to bounce ideas off from time to time; and Cris Loop, Kate's mom, for reading through a draft of this course and previewing the optional videos, providing vital feedback from a homeschool mom's perspective.
- The team at Master Books for making publishing this possible and for doing so with excellence. We especially want to thank Jennifer Bauer, the graphic designer, for making the material come to life, designing illustrations to help make complex concepts simple. We were consistently delighted at how the material brightened after she worked on it. And she patiently persevered through months of edits and adjustments.
- All of our friends, family, and teachers over the years for contributing to our own development and/or encouraging us along during this project. We'd like to specifically thank the Wonderful Wednesday Women for their many prayers and support of Kate during the writing of this curriculum, as well as both of our parents for their support over the years.

Above all, we want to thank God, without whose enabling none of this would be possible.

With gratitude,
Katherine and Dr. Adam Hannon

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Principles of Algebra

About This Curriculum

This complete Algebra 2 program not only teaches algebra, but also shows students why they're learning concepts and how algebra's very existence points us to God. Students will see algebra in action . . . and find their biblical worldview built along the way.

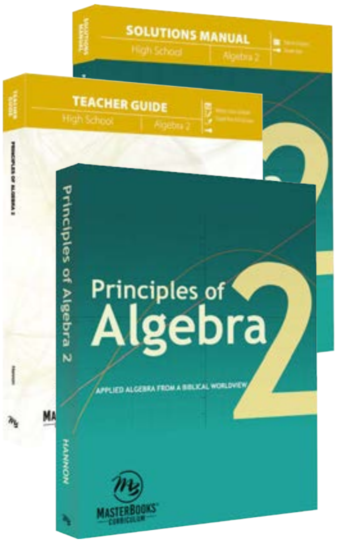
Are There Any Prerequisites?

At a minimum, students should have completed an Algebra 1 course. Completion of a full course in geometry is recommended, but not required.

Overall, students need to be familiar with the basics of algebra (working with algebraic fractions, exponents, and roots; combining like terms; basic factoring; formulas; basic finding of unknowns; and graphing of linear equations) as well as arithmetic (including converting units of measure, decimals, scientific notation, and using a calculator for exponents, roots, and operations inside of parentheses) and basic geometric ideas (such as perimeter, area, and volume of circles and simple polygons). A basic familiarity with these concepts is assumed.

How Do I Use This Curriculum?

This curriculum is designed so that it can be self-taught. Students should be able to read the material and complete assignments on their own, with a parent or teacher available for questions. The optional eCourse can limit the amount of reading and provide more guidance through the concepts. If teaching in a classroom, the text can serve as the basis for the teacher's presentations, with the text available as a reference later for students. This *Student Textbook* is divided into chapters and then into lessons. The number system used to label the lessons expresses this order. The first lesson is labeled 1.1 because it is Chapter 1, Lesson 1.



What Are the Curriculum's Components?

The curriculum consists of this *Student Textbook*, the *Teacher Guide*, and *Solutions Manual*. The *Student Textbook* contains the instructional lessons. The *Teacher Guide* contains an easy-to-follow schedule, as well as the worksheets, quizzes, and tests. The *Solutions Manual* contains a complete answer key (which includes solutions for most problems, as well as notes and explanations of many of the solutions).

Optional *Principles of Algebra 2 eCourse* available from Master Books Academy — These videos offer presentations of lessons produced by the author and is an addition to the printed material. They're great for students that are more visual/auditory in learning or need more walking through the concepts. Students using the videos should watch the video for that lesson then look over the text, studying it as needed. The eCourse is available through the Master Books Academy at MasterBooksAcademy.com.

Where Faith Grows!

Go online with Master Books video supplements!

What Do I Need To Complete This Course?

The supplies needed for this course are as follows:

- *Principles of Algebra 2: Applied Algebra from a Biblical Worldview Student Book*
- *Principles of Algebra 2 Teacher Guide*
- *Principles of Algebra 2 Solutions Manual*
- *Principles of Algebra 2 eCourse* (optional)
- Binder with Lined Paper or Other Note-Taking Method
- Calculator (or Online Calculator)
- A College Notebook
- A Second Notebook or Additional Lined Paper
- Index Cards (optional)
- Graph Paper

Please see page 6 of the *Teacher Guide* for a more complete description of each item.

Where Do I Go Upon Completion?

Upon completion of this course, students should complete a geometry course (if not already completed); if they have already completed geometry, students should be ready to begin a precalculus course or a college algebra program. Advanced students may also want to study for and take the *College Algebra* CLEP test. While this book does *not* cover everything on the test, advanced students may be able to use a CLEP study guide to fill in additional college-level concepts (as well as to gain familiarity with the test format).

Principles of Algebra

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Chapter 1

Setting the Foundation

1.1 God and the Laws of Math

Math — particularly upper-level math like algebra — has become associated in many people's minds with confusing rules and endless equations that seem disconnected from reality and pointless to learn.

$$TC(Q, q_i, m_i) = \sum_{i=1}^{n} \left[\frac{D_i}{m_i q_i} S_i + c_i^{v} D_i + \frac{q_i H_i^{v}}{2} \left(m_i \left(1 - \frac{D_i}{P_i} \right) - 1 + 2 \frac{D_i}{P_i} \right) \right]$$

But math — including algebra — is *not* disconnected from reality . . . and it's not pointless to learn! Regardless of your past experience in math, we invite you to join us on an exciting journey. Yes, we will have to grapple with rules and equations, but we'll do so while seeing how they apply outside of a textbook and how they proclaim the praises of the Creator.

Yes, you heard that right: Math declares praise to God. We live in a consistent universe — a universe so consistent, in fact, that we can record those consistencies and rely on them to hold true day after day, year after year. And that is exactly what we're doing in math! Every time you jump up, you come back down to the ground (we call that the law of gravity). Every morning the sun rises and in the evening it sets (the earth rotates around its axis once every 24 hours). Every time you plug a power cord into a wall outlet, you power its device (the laws of electromagnetism). All of these consistencies can be described using math.

For example, we can use letters to represent the consistent relationship between the force applied to an object (F), the velocity (v — think speed in a certain direction) of that object, and the power (P) produced: $P = Fv$. This relationship holds true over and over and over again because this universe operates in a predictable way, making modern science possible.

This relationship assumes the force is in the same direction as the velocity.

$$P = Fv$$

$$Power = Force\ (velocity)$$

Now why is the universe so consistent? Why do mathematical laws hold true? Because that's how God set up the universe, and day after day, year after year, He is continuing to keep His "covenant" with the "fixed order" around us — an order that math helps us describe.

> *Thus says the LORD: If I have not established my covenant with day and night and the fixed order of heaven and earth, then I will reject the offspring of Jacob and David my servant and will not choose one of his offspring to rule over the offspring of Abraham, Isaac, and Jacob. For I will restore their fortunes and will have mercy on them. (Jeremiah 33:25–26; ESV)*

Do you catch what God is saying here in Jeremiah? He's telling His people to look around and see how faithful He is to sustain the consistencies around them and telling them He'll be just as faithful to keep His promise to them too. The very consistencies around us — which math helps us record — serve as a testimony to what a faithful, covenant-keeping God we have! Math should continually remind us that we *can* trust God.

Are you beginning to catch a glimpse of how exciting math can be? Every problem you solve and new consistency you learn about is shouting out at you that God is still faithful, keeping His covenant with the "fixed order" . . . and it's that same faithful God who has promised to save all who call upon Him (Romans 10:13) and to complete the work He begins (Philippians 1:6).

Math is ultimately a way of describing the "fixed order" God put in place and sustains (by the word of His power, no less! (Hebrews 1:3)).

For years now, you've been adding, subtracting, multiplying, and dividing numbers. All of these processes are known as mathematical **operations**. When we add, we're really describing how God determined quantities to combine. When we multiply, we're really describing how God causes sets of quantities to combine (4 *times* 3 means 4 *sets of* 3). All of math is a way of describing the "fixed order" God made all around us.

You've also likely learned about various **properties** in math. Properties, like operations, are ways of describing the "fixed order" God created and sustains. The key properties of arithmetic are below as a review — make sure you know them, as we will continue to build on them throughout this book.

Note that because of the consistent way God holds all things together, we can rely on these properties to hold true for all numbers **and generalize the relationships using letters to stand for any number**.

Properties

Commutative Property of Addition and Multiplication

Order doesn't matter.

Addition	Multiplication
$a + b = b + a$	$ab = ba$

The *a* and *b* could stand for any number — we're just saying it doesn't matter which number comes first, the answers will be the same. Note that ab means a times b — when we're using letters to stand for numbers, we don't have to bother to write out a multiplication sign; just putting them next to each other means to multiply them.

Below is an illustration with actual numbers plugged in for the a and b placeholders.

Addition	Multiplication
$1 + 2 = 2 + 1$	$2(3) = 3(2)$
$3 = 3$	$6 = 6$

Note that the **parentheses** in 2(3) means to multiply. A number or symbol next to a parenthesis means to multiply by whatever is inside the parentheses.

Associative Property of Addition and Multiplication

Grouping doesn't matter. The grouping would be useful if there were different operations and we wanted the problem solved in an unusual order — but they don't affect anything when it's all addition or all multiplication, as we'll get the same answer no matter how we group numbers being added or multiplied.

Addition	Multiplication
$(a + \mathrm{b}) + c = a + (b + c)$	$(ab)c = a(bc)$

Again, here's an example with numbers plugged in to the placeholders:

Addition	Multiplication
$(1 + 2) + 3 = 1 + (2 + 3)$	$(2 \cdot 3)4 = 2(3 \cdot 4)$
$3 \quad + 3 = 1 + \quad 5$	$(6)4 = 2(12)$
$6 = 6$	$24 = 24$

Notice that we used several different ways of showing multiplication in this box — remember, ×, •, quantities right next to parentheses, and letters written next to each other all mean multiplication.

Identity Property of Addition and Multiplication

Adding 0 doesn't change the value.	*Multiplying by 1 doesn't change the value.*
Addition	**Multiplication**
$a + 0 = a$	$1a = a$
Example: $2 + 0 = 2$	*Example:* $1(2) = 2$

Properties of Division

Any Number (Except 0) Divided by Itself Equals 1.

$$a \div a = 1, \text{ provided } a \neq 0$$

Division by Zero

Strictly speaking, we can't divide by 0, as you can't divide something by nothing. So, if you encounter something like 10 divided by $a - 2$ (which would be written $\frac{10}{a-2}$), note that a cannot equal 2, or else we would be dividing by 0, since 2 – 2 equals 0.

In higher level math such as calculus, you will learn there are more nuanced ways of looking at these seemingly impossible divisions.

Math Is a Useful Tool

Because math helps us describe the "fixed order" God put in place around us, it's useful outside of a textbook! You may have noticed the word "applied" in the subtitle of this course. We want you to be equipped to use the concepts you learn outside of a textbook — to apply them.

Walter W. Sawyer compares mathematics to "a chest of tools" and urges students to know how to use them.

> *Mathematics is like a chest of tools: Before studying the tools in detail, a good workman should know the object of each, when it is used, how it is used, what it is used for.*[1]

Tools come in all sorts of layers of complexity. There are very simple tools, such as a hammer. And then there are tools like high-powered routers that serve a specific task. These latter tools take longer to learn how to use and serve a more specialized function, but they're very powerful for what they do.

Some of the concepts we'll be studying in this course are like that high-powered router. They are incredibly useful, but are more focused in their application. Thus you may not find yourself using all of them on a regular basis . . . or even at all, depending on the field you pursue. However, by learning how to use these high-powered tools, you'll both better understand the complexities of God's creation (and how the myriad of technological devices around us work) and be better equipped to think through other types of problems you might encounter. In other words, they can help you both grow in appreciation for the Creator and learn problem-solving and critical-thinking skills that can then be transferred to other areas of life and can help you complete the tasks God gives you to do. Above all, studying these tools will give you a deeper look at how math does, indeed, describe God's creation and proclaim God's praises.

Introducing Algebra and Formulas

This course is specifically an **algebra** course, meaning we're going to focus on the branch of math in which "letters and other general symbols are used to represent numbers and quantities in formulae and equations."[2] So you're going to see a lot of letters standing for quantities.

While we'll soon move on to not only using letters to stand for unknown quantities, then also determining the value of those "unknown quantities by means of those that are known,"[3] but we're also going to start in these first several chapters by reviewing the basics.

One basic application of algebra you've been using for years is that of allowing us to represent formulas. A **formula** is simply "a mathematical relationship or rule expressed in symbols."[4] You've been using formulas for years. For example, to find the circumference of a circle, you multiply the irrational number π, that begins 3.14, by the diameter.

$$Circumference = \pi(diameter)$$

Rather than writing all of those words out, we typically express this relationship as a formula. **Notice that we've used letters to stand for the circumference and distance!**

$$C = \pi d$$

To use the formula, we replace the letters with the values for that particular circle. For example, if we know a circle has a diameter of 4 in, we would have this:

$$C = \pi(4 \text{ in}) \approx 3.14(4 \text{ in}) \approx 12.56 \text{ in}$$

Notice that we used an **approximately equal sign** ($\approx$). We did that because we used a rounded value for π — thus the answer is not an exact answer. In this course, we'll use the approximately equal sign whenever we round just to make it clear we're using rounded values.

Note: We used an approximate value of 3.14 for π. When solving problems in this course, you can either use whatever rounded value you've memorized, or simply press the π button on your calculator to use the rounded value it has stored.

To use a formula, insert the appropriate values for the various letters and simplify!

As another example, suppose the diameter of a circular hot tub was 5.62 ft. In that case, we would find the circumference like this:

$$C = \pi(5.62 \text{ ft}) \approx 3.14(5.62 \text{ ft}) \approx 17.6468 \text{ ft} \approx 17.647 \text{ ft}$$

Since this is an applied algebra course, you may find yourself working with units of measure more than in your previous math courses. We'll walk through how to work with them as we go.

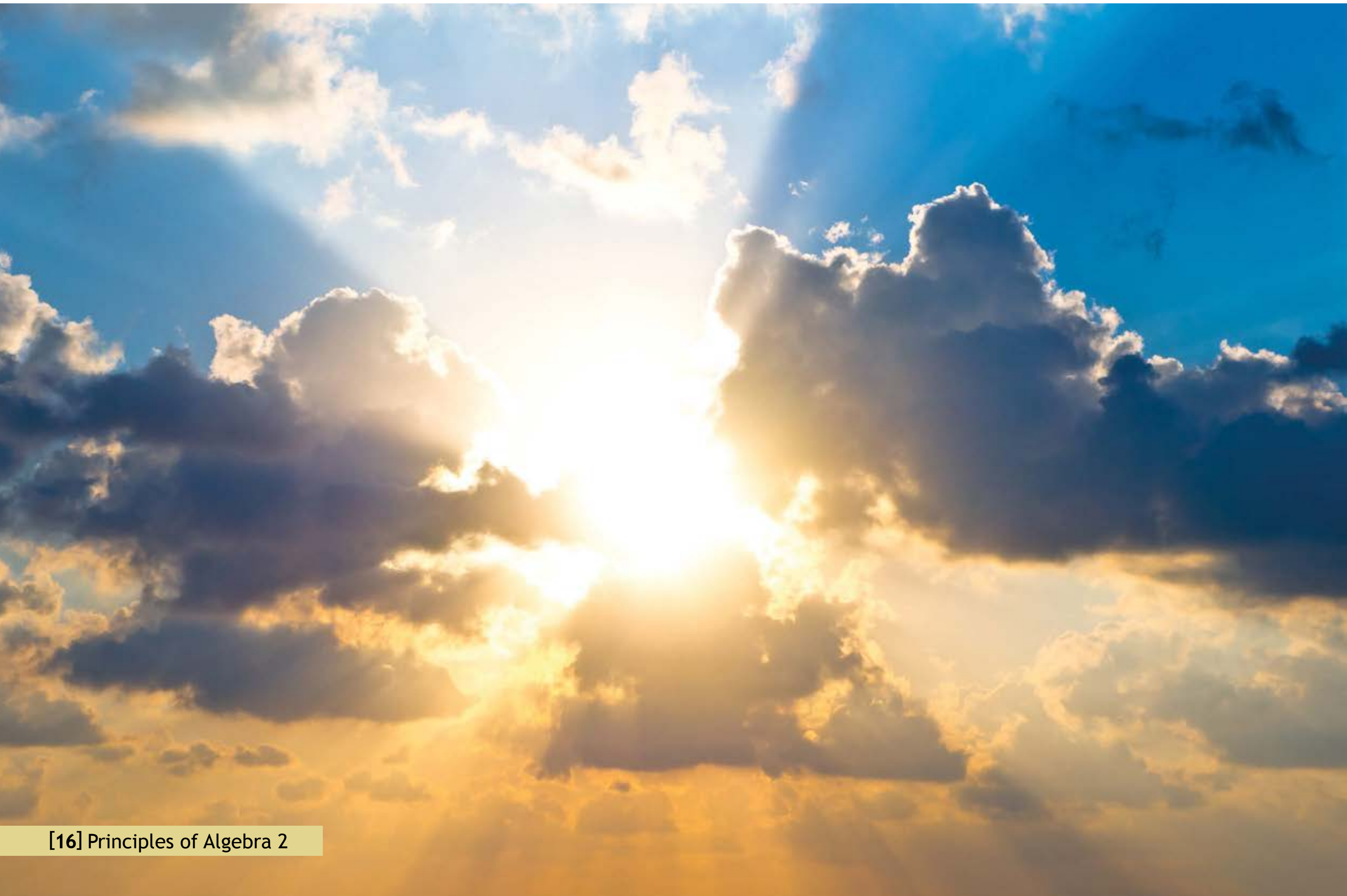
Notice that we rounded to the 3rd decimal place. In this course, **always round to the 3rd decimal place unless instructed otherwise . . . and use the ≈ to show that you did!** (If needed, see the endnote for a reminder on how to round.[5])

Also notice that we included the unit of measure in the answer. **Always include units of measure in the answer if one is given.** Why? Because otherwise no one knows what the answer really is. 17.647 could be 17.647 centimeters, pounds, $ ... we need units to know what is being represented!

Keeping Perspective

As we embark on an exploration of math in this course, always remember that math is simply a way of describing the "fixed order" God created and sustains.

Above all, remember that math is shouting out at you that just as faithful as God is to keep His covenant with the "fixed order" of creation, He will be just as faithful to everything else He's said in His Word, the Bible. This should give you incredible peace if you've placed your trust in His way of salvation — Jesus — as God then promises to save and sanctify you. But it should give you great fear if you're trusting in anything else for salvation, as God says that no one comes to God apart from Jesus (John 14:6).[6] Make sure your trust is in Jesus (see Appendix A: Math's Message). Then get ready for an exciting journey into exploring His "fixed order."



1.2 The Language of Mathematics

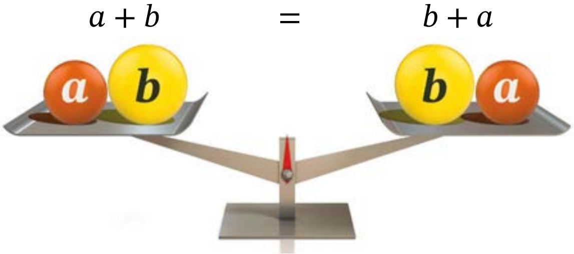
If we're going to describe the "fixed order" around us, we need a language to use! And just like learning any spoken language takes work, so too does it take work to learn the language of mathematics. But once you know the language, it makes communication simple. We can think of the language side of mathematics as a **convention**, or "a way in which something is usually done, especially within a particular area or activity."[7]


connected,
communicāre,
mu'ni•ca'tor n.
com•mu•ni•ca•tion
communicating; trans
messages, or informat

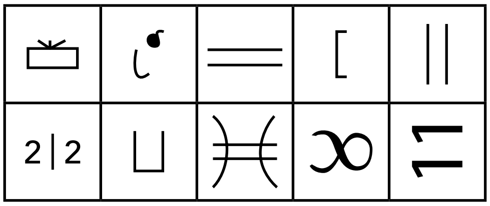
Just as words help us describe quantities and concepts, so can symbols. In math, though, we prefer symbols, since it's a lot easier to work with them. (Imagine trying to add a million, four hundred thirteen thousand, seven hundred nineteen with four thousand, three hundred ninety without first rewriting it using symbols as 1,413,719 + 4,390.)

We have symbols to stand for quantities with a known value, such as the symbol 1, 0.56, π, etc. We also have symbols to stand as placeholders for unknown quantities, such as a, v, V, σ, etc. Notice that when we use a letter, **the case matters**. V is representing a different value than v. (In fact, you might encounter them both in the same problem, with V standing for volume and v for velocity.)

You've already learned a lot of symbols — the plus and minus signs, different ways to multiply (see box), etc. You know that an **equal sign** (=) is an agreed upon way to represent that the quantities on both sides of the sign have the same exact value.



This symbol, though, is really just a convention for representing the consistency God created and sustains. Note that we could have used different symbols to show equality — in fact, below are a few that have historically been used.[8] The symbol is simply part of the language system we're using to communicate about the consistencies around us.



Sometimes we want to compare quantities that are not equal. In that case, we're working with an **inequality**. And you've already learned various symbols to help you describe inequalities.

Hint: When using the less than and greater than signs ($<$ and $>$), put the smaller side of the sign with the smaller quantity.

6 does not equal 7;
$6 \neq 7$

6 is less than 7;
$6 < 7$

7 does not equal 6;
$7 \neq 6$

7 is greater than 6;
$7 > 6$

Different Symbols for Showing Multiplication

Using a × or • sign or putting quantities right next to parentheses are all ways of showing multiplication. The expressions below all mean 5 *times* 2.

$$5 \bullet 2 \qquad 5 \times 2 \qquad 5(2)$$

Note: Since the letter x is used often for unknowns in algebra and is hard to distinguish from × when handwritten, be careful when using × to mean multiplication. Parentheses is often a better choice in algebra.

If we're using letters to stand for quantities, we don't need to include a multiplication sign (***ab* means *a* times *b***, and $5b$ means 5 times b). This is an agreed-upon convention to simplify expressing multiplication.

When exploring the language side of mathematics, it's important to keep in mind that symbols and other conventions can (and sometimes do!) vary. Just like language systems have varied since the Tower of Babel, the way we express mathematical concepts also varies (Genesis 11). Much of math consists of conventions: agreed-upon protocols or rules that aid us in communication. Man is only able to develop conventions because God made us in His image, capable of subduing the earth (Genesis 1:27).

Symbols for Units of Measure/Conventions in Showing Units

As we apply algebra, we'll encounter numbers that have units of measure quite frequently. After all, if we're weighing something, we need to know if that weight is in pounds, newtons, or some other unit.

There are many different conventions regarding how to write units of measure. Appendix B: Reference Section lists many of the units of measure you'll encounter and the abbreviations used in this course. It's worth noting, though, that those abbreviations can (and do!) vary. For example, while we'll abbreviate seconds as s, you may see it abbreviated elsewhere as sec. The abbreviations we choose are just conventions.

It's also worth noting that **we work with units of measure much the same way as we do with letters we're using to stand for quantities**, only we need to be careful to treat the whole abbreviation as the unit. For example, we view kg as standing for kilograms, *not* as a k multiplied by a g.

Note that in this course, we will italicize letters standing for unknowns, but **we will not italicize letters that are a unit of measure.**

Let's say we were to multiply 7 kg by 3 m. What do we do with the units of measure? We multiply them!

$$(7\ \mathrm{kg})(3\ \mathrm{m}) = 21\ \mathrm{kg} \bullet \mathrm{m}\ \textit{or}\ 21\ \mathrm{kg\ m}\ \text{(pronounced “21 kilogram-meters”)}$$

Notice that with units, we didn't just put the two units next to each other (i.e., we didn't write kgm) like we would when using unknowns. It would be too easy then to think the entire unit was kgm, or that we meant k times g times m, rather than that we have a unit of kg *times* m. Some resources, though, will just put a space between the two units and leave them next to each other, like 21 kg m. Either way clearly shows the unit of measure.

Now you might be wondering what a "kilogram-meter" is. Often the units we come to in problems don't have a specific definition like a kilogram or a meter does. In this case, a kilogram-meter is simply the result of multiplying kilograms by meters. (We can use it to measure the work done by applying a force, such as to a pump.[9])

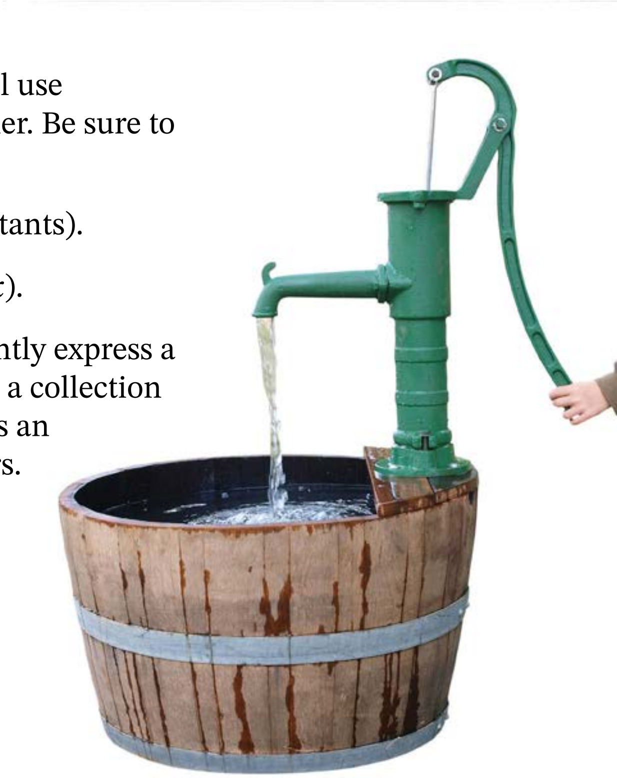
Words Used in Math

Speaking of the language side of math, here are a few words we'll use extensively as we explore the principles of applied algebra together. Be sure to familiarize yourself with them before we get started.

Constants – Quantities that have a fixed value (5 and π are constants).

Variables – Values whose value can vary in a problem (such as x).

Expression — An expression is "a collection of symbols that jointly express a quantity."[10] For example, $4 + 5$ is an expression — 4, 5, and + are a collection of symbols that together express the quantity 9. Likewise, $a + b$ is an expression. The a and b here are simply placeholders for numbers.



Equation — An equation is “a statement that the values of two mathematical expressions are equal (indicated by the sign =).”[11]

Simplify — When we refer to simplifying an expression or equation, we mean to express it as simply as possible. For example, 5 + 6 simplifies to 11.

We’ll explore the definition of constants and variables further in Chapter 7.

Unless instructed otherwise, **simplify your answers as much as possible** in this course.

Keeping Perspective

On your worksheet today, you’re going to review some conventions we’ll be following in this course. Make sure you’re familiar with them so that you can follow along as we go forward. And remember to be patient with yourself as you try to learn the language side of mathematics — it takes work to learn another language!

1.3 Understanding, Multiplying, and Dividing Fractions

Throughout this first chapter we're going to be reviewing some common foundational conventions. Most of these conventions should be familiar, but we are hoping you will see them in a new light as we both review them *and* see how they apply outside of a textbook. We're going to start with fractions.

Understanding Fractions

A lot of the confusion stems from the fact that the word "fraction" is used in different ways. A fraction can refer to

- a partial quantity, no matter how it's written. *Examples:* $\frac{1}{4}$, 0.25, 25%
- a specific notation with a **numerator** (top number) showing the number of parts and a **denominator** (bottom number) showing the parts in a whole. *Examples:* $\frac{1}{4}, \frac{8a}{c}, \frac{5+d}{5}$

Historical Tidbit

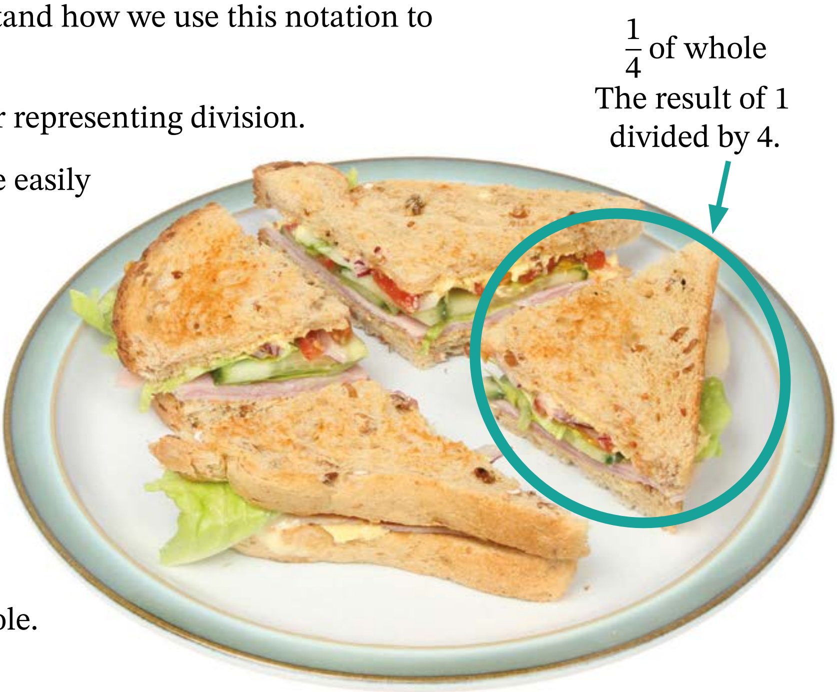
Did you know that fractions at one point were written without a line? So $\frac{4}{5}$ would have been $\genfrac{}{}{0pt}{}{4}{5}$ instead. Leonardo Pisano "was one of the first to separate the numerator from the denominator by a fractional line."[12] Remember, while God created the real-life quantities and consistencies fractions describe, the notations and conventions used to describe them have (and still do!) vary. God made man in His image with creativity to develop new ways of exploring and describing His creation.

However, there's a third meaning to the word "fraction," and it's this third meaning that will help you really understand how we use this notation to describe God's creation.

- Fractions are simply a convention for representing division.

If I want to write 1 ÷ 4, I can do this more easily using a fraction line: $\frac{1}{4}$.

Now you may be used to thinking of $\frac{1}{4}$ as one part out of four . . . and it is. If you take 1 sandwich and you divide it into 4 pieces, each piece is $\frac{1}{4}$ of the whole. The $\frac{1}{4}$ here could be thought of as the division problem *or* as the number of parts over the number of parts in the whole.



$\frac{1}{4}$ of whole

The result of 1 divided by 4.

Understanding that the fraction line can be thought of as a division symbol is critical for algebra, as we typically won't be using a division sign — just the fraction line. If we want to write $5a \div b$, we will simply write $\frac{5a}{b}$. Likewise, to write $9 \div 2$, we could write $\frac{9}{2}$. Notice that this notation is more concise. It proves quite useful.

We call $\frac{9}{2}$ an **improper fraction** because its numerator (9) is greater than its denominator (2). While in other math courses you might have rewritten this as a **mixed number** (a number with a whole part and a fractional part — $4\frac{1}{2}$ in this case), in algebra **we avoid mixed numbers.** The reason is that in algebra, we work a lot with letters, which get written right next to numbers to show multiplication. Thus it'd be easy to accidentally mistake $4\frac{1}{2}$ for 4 times $\frac{1}{2}$ rather than as 4 wholes and $\frac{1}{2}$. In this course, **do not use mixed numbers.**

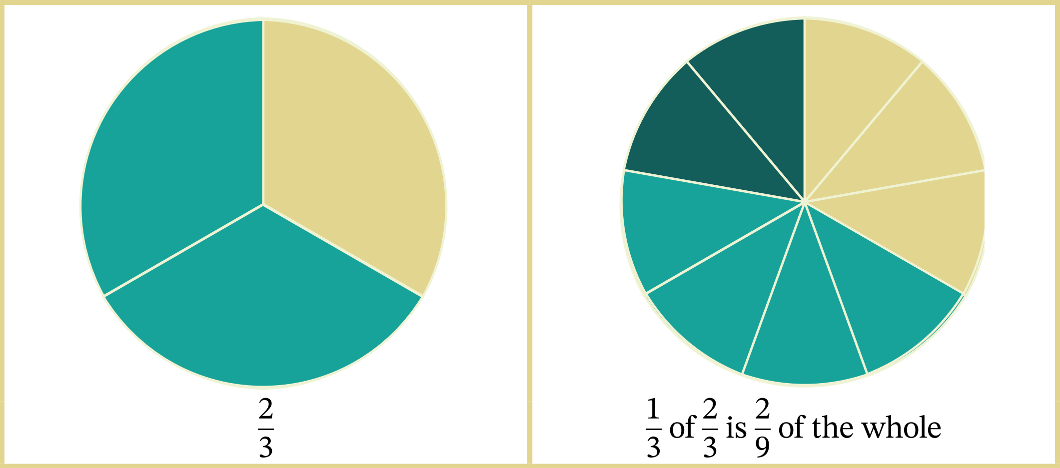
Keeping this in mind, let's briefly review how to multiply and divide fractions. It's important that you thoroughly know how to work with them, whether dealing with known quantities or with letters standing for unknown quantities.

Multiplying Fractions

We're used to thinking of multiplication in terms of repeated addition. If we want to figure out how much we'll pay if each ticket costs $2 and we want 3 of them, we multiply 3 times $2 to add up the cost of 3 tickets. But when we multiply by a fraction with a value *less than 1*, we're finding a quantity *of* another quantity. For example, $\frac{1}{3}\left(\frac{2}{3}\right)$ is finding a third *of* two thirds. We might need to find this if we were thirding a recipe that called for $\frac{2}{3}$ cup of flour and trying to figure out how much flour to put in instead.

How do we actually perform the multiplication? **We multiply fractions by multiplying the numerators together and the denominators together.**

$$\frac{1}{3}\left(\frac{2}{3}\right) = \frac{1(2)}{3(3)} = \frac{2}{9}$$



Notice that we can multiply the numerators and denominators when we have used a letter(s) to stand for an unknown value(s) too. Why? Because of the consistent way God governs all things!

Example: $\frac{a}{5c}\left(\frac{3}{b}\right) = \frac{a(3)}{5c(b)} = \frac{3a}{5bc}$

In the above example, note how we listed the 3 and the 5 first in our final answer, making the numerator $3a$ instead of $a3$. Also notice that we wrote $5cb$ as $5bc$. Except in some science or application situations, **it's standard to list constant values in a term first, then all unknown quantities (represented by letters), in alphabetical order.**

Sometimes, we need to multiply a fraction by a whole number. Once again, if the fraction is less than 1, we're finding a portion *of* another quantity. For example, $\frac{1}{3}(30)$ is $\frac{1}{3}$ of 30. If we want to figure out what $\frac{1}{3}$ of a class of 30 college students is, we'd multiply $\frac{1}{3}(30)$.

On the flip side, if we wanted to multiply a recipe by 30 to make enough for a crowd and the recipe calls for $\frac{1}{3}$ c of flour, we'd be finding $30\left(\frac{1}{3}\right)$ . . . which is finding repeated addition (adding $\frac{1}{3}$ thirty times).

And how do we complete the multiplication in either case? Well, since any number divided by 1 equals itself, we can think of 30 as $\frac{30}{1}$. Thus, we can multiply a fraction by whole numbers the same way we would fractions: just view the whole number or unknown as a numerator of a fraction with a 1 as the denominator.

$$\frac{1}{3}(30) = \frac{1}{3}\left(\frac{30}{1}\right) = \frac{30}{3} = 10$$

($\frac{1}{3}$ of a class of 30 students is 10 students)

$$30\left(\frac{1}{3}\right) = \frac{30}{1}\left(\frac{1}{3}\right) = \frac{30}{3} = 10$$

(taking $\frac{1}{3}$ c of flour 30 times results in 10 c of flour)

It's not necessary to write out all of the steps shown in this first example. They're just here to help clarify.

Example: $5\left(\frac{a}{b}\right) = \frac{5}{1}\left(\frac{a}{b}\right) = \frac{5a}{1b} = \frac{5a}{b}$

Example: $4ac\left(\frac{b}{d}\right) = \frac{4abc}{d}$

Notice that we listed the letters in alphabetical order. $4acb$ and $4abc$ mean the same thing, as multiplication is commutative and associative.

Example: $x\left(\frac{y}{z}\right) = \frac{x}{1}\left(\frac{y}{z}\right) = \frac{xy}{z}$

We explored the "rules" of working with fractions — such as inverting and multiplying — much more in *Principles of Mathematics, Book 1*. We're just reviewing them briefly here, but please see the earlier book to see how each rule really does rest on the consistent way God causes objects to operate. Also note that if the numerator and the denominator fully divide by the value we're dividing by as in $\frac{32}{25} \div \frac{8}{5}$, we can simply divide the numerators and the denominators: $\frac{32 \div 8}{25 \div 5} = \frac{4}{5}$. The "invert and multiply" rule comes in handy, though, when the numerators or the denominators don't divide evenly, as in $\frac{1}{5} \div \frac{2}{3}$. Were we to simply divide the numerators and denominators here, we'd end up with $\frac{1 \div 2}{5 \div 3}$, *or* $\frac{\frac{1}{2}}{\frac{5}{3}}$, which isn't very simplified.

Dividing Fractions

To divide by a fraction, we invert (i.e., take the multiplicative inverse of) and multiply. This "rule" is a shortcut to help us quickly find an answer. There is another way to do it . . . only this is simpler. And again, we can do it with either known or unknown quantities.

$$\frac{1}{2} \div \frac{2}{3} = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

inverted and multiplied

$$\frac{x}{b} \div \frac{2a}{3} = \frac{\frac{x}{b}}{\frac{2a}{3}} = \frac{x}{b}\left(\frac{3}{2a}\right) = \frac{3x}{2ab}$$

inverted and multiplied

Let's think through for a minute what we're really finding with division. Division is "the action of separating something into parts or the process of being separated."[13] This holds true for fractions as well, only it's important to realize that when we divide by a fraction with a value less than 1, we end up with a *greater value.* For example, $30 \div \frac{1}{4} = 30\left(\frac{4}{1}\right) = 120$. One example of this is if we have \$30 and divide it up into quarters where each quarter is $\$\frac{1}{4}$ (i.e., 25 cents = \$0.25 = $\$\frac{1}{4}$), we'd get 120 quarters.

Or consider the example we looked at under multiplication with thirding a recipe. If we need to third a recipe calling for $\frac{2}{3}$ c flour, we can either look at this as finding $\frac{1}{3}$ of $\frac{2}{3}$, or $\frac{1}{3}\left(\frac{2}{3}\right) = \frac{2}{9}$, or as dividing $\frac{2}{3}$ by 3. $\frac{2}{3} \div 3 = \frac{2}{3}\left(\frac{1}{3}\right) = \frac{2}{9}$. We get the same answer either way. And notice that, as with multiplication, **we viewed the whole number (the 3) as $\frac{3}{1}$ and then inverted and multiplied.**

Multiplicative Inverse Review

Multiplicative inverse is a name to describe the number that, when multiplied by another number, equals 1. Some people also call it the **reciprocal** of a number, or, so long as the context is clear, simply the inverse. **In a fraction, the denominator becomes the numerator and the numerator the denominator.**

$\frac{7}{2}$ is the multiplicative inverse of $\frac{2}{7}$, as $\frac{7}{2}\left(\frac{2}{7}\right) = \frac{7(2)}{2(7)} = \frac{14}{14} = 1$

$\frac{b}{3}$ is the multiplicative inverse of $\frac{3}{b}$, as $\frac{b}{3}\left(\frac{3}{b}\right) = \frac{3b}{3b} = 1$

(As we saw in Lesson 1.1, any quantity except 0 divided by itself equals 1. Note that here we're assuming b does not equal 0; $\frac{3b}{3b}$ can't be simplified if $b = 0$.)

Remember, the fraction line means division.

So $\frac{\frac{4}{5}}{\frac{2}{7}}$ means $\frac{4}{5}$ divided by $\frac{2}{7}$.

Example: $\frac{\frac{1}{5}}{\frac{2}{7}} = \frac{1}{5} \div \frac{2}{7} = \frac{1}{5}\left(\frac{7}{2}\right) = \frac{1(7)}{5(2)} = \frac{7}{10}$

Example: $\frac{\frac{a}{5}}{\frac{3}{b}} = \frac{a}{5}\left(\frac{b}{3}\right) = \frac{ab}{5(3)} = \frac{ab}{15}$

To keep things simple to grade, if a problem is given to you in fractional form and there's a fractional part in the answer, give your answer as a fraction. If there's a decimal in the problem and there's a fractional part in the answer, then give the answer using a decimal. If a problem has both fractions and decimals, you can give your answer in either form.

Keeping Perspective

As you review fractions, remember that we work with them because they help us describe God's creation. And the various methods and "rules" help us accurately describe the real-life principles God created and holds together.

1.4 Equivalent Fractions and Simplifying Fractions

It's time now to continue reviewing fractions by exploring the concept of equivalent fractions and simplifying fractions. Both of these concepts should already be familiar to you, but they're so important that they are worth a review. Not only does simplifying fractions help us better represent information in an easier-to-process way (it is easier to instantly know what quantity is meant by $\frac{1}{3}$ than by $\frac{30}{90}$), but it also helps us add and subtract fractions . . . and even find unknown quantities! In short, it's a skill we'll be using a lot.

Equivalent Fractions

Recall from Lesson 1.1 that multiplying by 1 doesn't change the value (the identity property of multiplication). Since any number divided by itself equals 1, then $\frac{8}{8}$ or $\frac{a}{a}$ are fractions worth 1 (remember, $\frac{8}{8}$ means 8 divided by 8, and $\frac{a}{a}$ means a divided by a).

Even though we don't know a's value, we know that we have some value divided by that same value. Because of the consistent way God governs all things, we know the answer will be 1 (as long as $a \neq 0$).

It follows **that if we multiply a fraction by a fraction equal to 1, we're not changing the value.** Instead, we're forming what we call an **equivalent fraction**.

Example: $\frac{4}{2}\left(\frac{8}{8}\right) = \frac{4(8)}{2(8)} = \frac{32}{16} = 2$

Notice that multiplying by $\frac{8}{8}$ didn't change the value. $\frac{4}{2}$ (which means 4 *divided by* 2) equals 2, as does $\frac{32}{16}$ (which means 32 *divided by* 16).

Because of the consistent way God governs all things, this holds true for unknown quantities as well.

Example: $\frac{c}{8b}\left(\frac{a}{a}\right) = \frac{ac}{8ab}$

Note that we wrote the numerator as ac instead of ca and $8ab$ instead of $8ba$. Since multiplication is commutative and associative, the order and grouping doesn't matter. We put the a first, though, as it's a convention to list the letters in alphabetical order. Notice how it is easier to read $\frac{ac}{8ab}$ than in $\frac{ca}{8ba}$.

Knowing this can help us both simplify and add and subtract fractions, as we'll soon review.

Whole Numbers and Equivalent Fractions

We can form equivalent fractions for whole quantities too!

Example: $3\left(\frac{2}{2}\right) = \frac{3(2)}{2} = \frac{6}{2}$

Notice that $\frac{6}{2}$, which represents 6 *divided by* 2, does indeed equal 3.

Notice also that we viewed the 3 as a numerator. Since any number divided by 1 equals itself, we could have rewritten 3 as $\frac{3}{1}$ to clarify this.

$$3\left(\frac{2}{2}\right) = \frac{3}{1}\left(\frac{2}{2}\right) = \frac{3(2)}{2} = \frac{6}{2}$$

The same thing applies for unknowns as well.

Example: $a\left(\frac{x}{x}\right) = \frac{a}{1}\left(\frac{x}{x}\right) = \frac{ax}{x}$

Simplifying Fractions

We often need to simplify fractions. A simple way to think about this is to think about completing some of the division ahead of time.

Remember, the fraction line means to divide. So, if we can complete part of that division, that helps simplify the fraction.

You've already been doing this for years. For example, when you've seen problems like $\frac{18}{4}$, you see that both the numerator and the denominator can be divided by 2. So, you go ahead and complete that part of the division, simplifying the fraction down to $\frac{9}{2}$.

We could also think of this as dividing both the numerator *and* the denominator by 2.

$$\frac{18 \div 2}{4 \div 2} = \frac{9}{2}$$

Another way of thinking about it is that we're looking at the factors that make up the number and then seeing what is repeated in the numerator and denominator.

$$\frac{18}{4} = \frac{9 \bullet 2}{2 \bullet 2}$$

Notice we can then see that there's a 2 in both the numerator and the denominator. This is really a fraction worth 1!

$$\frac{18}{4} = \frac{9 \bullet 2}{2 \bullet 2} = \frac{9}{2}\left(\frac{2}{2}\right)$$

And since multiplying by 1 doesn't change the value, it follows that we can simply remove the $\frac{2}{2}$.

$$\frac{18}{4} = \frac{9 \bullet 2}{2 \bullet 2} = \frac{9}{2}\left(\frac{\cancel{2}}{\cancel{2}}\right) = \frac{9}{2}$$

$\frac{18}{4}$ and $\frac{9}{2}$ represent the same amount. Notice that we didn't have to rewrite the whole fraction; we could have just crossed out the 2 in the numerator and denominator.

$$\frac{18}{4} = \frac{9 \bullet \cancel{2}}{2 \bullet \cancel{2}} = \frac{9}{2}$$

Simplifying fractions becomes super important when we're dealing with unknowns, as it helps us solve problems we couldn't otherwise. For example, suppose we had $\frac{ac}{8a}$ and we knew the value of c but did *not* know the value for a. What can we do? Well, we know that, provided a is not 0, $\frac{a}{a}$ equals 1, as, due to the consistent way God governs all things, any number except 0 divided by itself equals 1. So, we can go ahead and complete that part of the division by crossing the a's out.

Unless told otherwise, you can assume when simplifying fractions in this course that the expression in the denominator does not equal 0.

$$\frac{ac}{8a} = \frac{\cancel{a}c}{8\cancel{a}} = \frac{c}{8}$$

Another way to think about what we just did is dividing both the numerator and the denominator by the *same amount.*

$$\frac{ac \div a}{8a \div a} = \frac{c}{8}$$

We can also think of it as removing the fraction worth one, since we know that won't affect the value.

$$\frac{ac}{8a} = \frac{c}{8}\left(\frac{a}{a}\right) = \frac{c}{8}$$

Be careful! You can only complete some of the division in a fraction when you're dealing with multiplication. If you had $\frac{\mathrm{a} + \mathrm{c}}{8\mathrm{a}}$ instead of $\frac{\mathrm{ac}}{8\mathrm{a}}$, you couldn't simply cross out the a's, as you're being asked to add a and c and *then* divide that amount by $8a$. It's only when you can view the numerator and denominator as a product of factors (i.e., you're dealing with multiplication!) that you can cancel out.

Keeping Perspective

Notice how in this lesson we built on the identity property of multiplication to help us form equivalent fractions and simplify them. As we dig deeper into math, we will be continuing to build. But always remember that everything rests on those consistencies God created and sustains!

Unless instructed otherwise, always simplify fractions in this course as much as possible. $\frac{10}{4x}$ should be simplified to $\frac{5}{2x}$ in your answer. (We divided both the numerator and the denominator by 2.) Simplifying makes it easier to see what's really going on at a glance and, as we mentioned at the beginning, will prove invaluable as we go forward. You'll get a chance on your worksheet to see some sample meanings for simplifying fractions, including seeing what portion of a city's population strongly supports a candidate.

1.5 Understanding Ratios and Proportions

One common application of fractions is to show ratios and proportions. As we explore these, we're going to get a chance to apply all we've looked at so far regarding fractions!

Ratios

A **ratio** is "the relative size of two quantities expressed as the quotient of one divided by the other."[14] In other words, "ratio" is a fancy name for using division to compare quantities! And since fractions represent division, they make a convenient notation to use to represent ratios.

If there are 25 students in a college class *per* only 5 computers, then the ratio between students and computers can be expressed as a fraction like this: $\frac{25\text{ students}}{5\text{ computers}}$. Note that we can simplify this fraction by completing the division, giving us $\frac{5\text{ students}}{1\text{ computer}}$, which we would read as 5 students *per* 1 computer.

There are other notations that can be used to express ratios. For example, we can write 25:5 instead of $\frac{25\text{ students}}{5\text{ computers}}$. But we'll use a fraction in this course.

Typically, we don't bother to write out a 1 before a unit of measure (such as computers). We would write $\frac{5\text{ students}}{\text{computer}}$ and read it as 5 students per computer.

If you take 2 vitamins *per* day, we could write a ratio between the number of vitamins like this: $\frac{2\text{ vitamins}}{\text{day}}$ or $\frac{2\text{ vitamins}}{1\text{ day}}$.

Or say you're making a recipe, and you know that you need 2 tsp of cumin *per* 4 c of water. We could write the ratio between the cumin and water like this: $\frac{2\text{ tsp}}{4\text{ c}}$.

Notice that we kept italicizing *per* up above. **If you can word a problem using the word *per*, it's a good indicator that you're probably dealing with a ratio!** Remember, a ratio is simply a comparison via division . . . which you can use a fraction to represent.

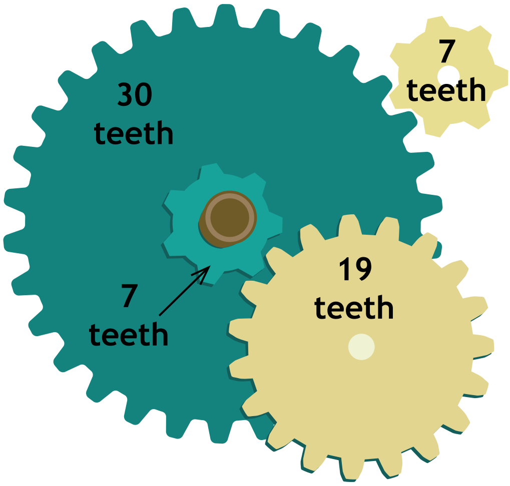
When written as a fraction, you can work with ratios just like you would with fractions!

Applying Ratios (and Multiplying Fractions, too!)

Gear Ratios

Let's look at an application of ratios. Along the way, we'll also apply multiplying fractions! Consider the gears in the engine shown on the next page.

Notice how the small gear on the right has 7 teeth, or spikes, while the large gear in the middle has 30. The ratio between them is $\frac{7}{30}$, or 7 *per* 30. Let's say



that we wanted to know what portion of the large gear rotated each time the small one rotated. Notice the use of the word *of*. We need to use multiplication! We can find that by multiplying the rotation of the small gear by the rotation of the large gear. The small gear is making a complete rotation, so it's going around 1 time. The large gear, though, is only going to make it through 7 out of its 30 teeth, as that small gear only has 7 teeth (and it is the small gear's teeth rotating that cause the large gear's teeth to rotate). Thus, we'd have this:

$$1\left(\frac{7}{30}\right) = \frac{7}{30}$$

Each time the small gear goes around 1 time, the large gear is going to go $\frac{7}{30}^{\text{th}}$ of the way around, as the small gear will push it through 7 out of its 30 teeth.

Now the middle gear that only has 7 teeth is attached to the large gear via a rod rather than being moved by its teeth. Each time the large gear goes around once, it goes around once as well. That also means that each time the initial small gear goes around once, the middle gear with 7 teeth only goes around $\frac{7}{30}^{\text{th}}$ of the way around, as it's moving via a rod attached to the large gear.

The final gear has 19 teeth. So each time the middle gear with 7 teeth goes around 1 time, the gear with 19 teeth will go around $\frac{7}{19}^{\text{th}}$ of the way, as the gear with 7 teeth will turn it through 7 of its 19 teeth.

$$1\left(\frac{7}{19}\right) = \frac{7}{19}$$

Keeping all this in mind, what portion *of* the 19-tooth gear rotates each time that initial 7-tooth gear rotates? Notice that we're trying to find the portion *of* something. The word *of* is a good indicator we need to use multiplication. And we do! We just say that every time the initial 7-tooth gear rotates, the middle 7-tooth gear goes $\frac{7}{30}$ of the way around, as it's attached via a rod to the large gear . . . and that each time it rotates, the final 19-tooth gear goes $\frac{7}{19}$ of the way around. So now we need to find $\frac{7}{30}$ *of* $\frac{7}{19}$, or $\frac{7}{30}\left(\frac{7}{19}\right)$.

$$\frac{7}{30}\left(\frac{7}{19}\right) = \frac{49}{570}$$

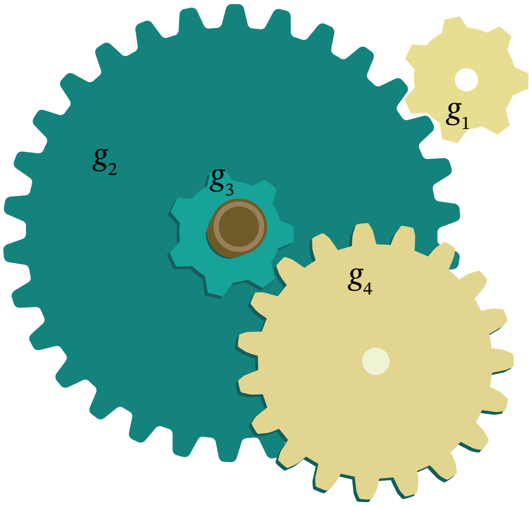
Now we've found what is called the gear ratio of the whole system: a comparison via division that, in this case, shows us the portion of the final gear that rotates each time the initial gear makes one rotation.

And since God governs a consistent universe, we could use letters to stand for the various gears and show how to find the combined gear ratio (represented by an *R*) for any size gear in this arrangement.[15]

$$R = \frac{g_1}{g_2}\left(\frac{g_3}{g_4}\right)$$

This in turn could be simplified to this:

$$R = \frac{g_1 g_3}{g_2 g_4}$$



Now we have a formula we can use to help us find the gear ratio for any gear!

Our point here is not to fully understand gears (so don't worry if you didn't follow all the details), but rather to show an application of what we reviewed today. Knowing the gear ratio helps in designing engines . . . and fractions, multiplying fractions, and ratios help us in the process!

Conversion Ratios

One common application of ratios is in converting from one unit to another. If something takes 1,800 seconds, what portion of an hour is that? We can find that like this:

$$1{,}800\ \cancel{\text{s}}\left(\frac{1\ \cancel{\text{min}}}{60\ \cancel{\text{s}}}\right)\left(\frac{1\ \text{hr}}{60\ \cancel{\text{min}}}\right) = 0.5\ \text{hr}$$

> Notice that min does not mean m times i times n. It is an abbreviation standing for "minute" and is treated as a single unit.

What did we do? Well, we multiplied by $\frac{1\text{ min}}{60\text{ s}}$. This is really a fraction worth 1, as both 1 min and 60 s represent the *same time*. When we convert units, we multiply by fractions worth 1 so as to not change the value (we're multiplying both the numerator and the denominator by the *same* time, distance, weight, etc., only expressed in different units). These are known as **conversion ratios**.

> Notice that the units canceled out when we had one in the numerator and one in the denominator like unknowns did in the last lesson. **We can work with units of measure the same way we do with unknowns**, multiplying them, dividing them, etc. So when we multiplied to convert 1,800 s, we crossed out units that were the same in both the numerator and the denominator, as they would cancel each other out.

Proportions

Connected with the idea of a ratio is that of a **proportion**, which is a fancy name for 2 equal ratios. In other words, there are 2 equivalent fractions, each of which represents a ratio.

For example, a friend who used to work at a beverage company once told us that soda companies have carefully guarded recipes for how to make their beverages . . . and about how he had to use algebra extensively to scale the recipes. Say one recipe calls for $\frac{1}{4}$ cup lemon juice *per* 1 gallon of water. Notice that this could be written as a ratio: $\frac{\frac{1}{4}\text{ c}}{1\text{ gal}}$. If they want to make a batch with

Again, notice the use of the word *per*. Remember that if you can insert *per* into the problem, it's a good indicator that you're working with a ratio.

30 gallons of water instead, how many cups of lemon juice will they need? We'd have this proportion, where x represents the cups of lemon juice needed.

$$\frac{\frac{1}{4}\text{ c}}{1\text{ gal}} = \frac{x}{30\text{ gal}}$$

We can figure out the cups of lemon juice needed by figuring out what value would be needed to make these ratios equivalent. Notice that 30 gallons is 30 times greater than 1 gallon, as $30 \div 1 = 30$. So x must also be a value that is 30 times greater than $\frac{1}{4}$ c! Let's multiply both the numerator and the denominator of $\frac{\frac{1}{4}\text{ c}}{1\text{ gal}}$ to form an equivalent fraction with 30 gallons in the denominator.

$$\frac{\frac{1}{4}\text{ c}}{1\text{ gal}}\left(\frac{30}{30}\right) = \frac{\frac{30}{4}\text{ c}}{30\text{ gal}} = \frac{\frac{15}{2}\text{ c}}{30\text{ gal}}$$

We would need $\frac{15}{2}$ cups.

Notice that to figure out what value to multiply $\frac{\frac{1}{4}\text{ c}}{1\text{ gal}}$ by in order to form an equivalent ratio with 30 gal in the denominator, we divided 30 gal by 1 gal. This told us that the denominator was 30 times greater, so we needed to multiply the numerator by the *same amount* so that we'd be multiplying by a fraction worth 1, thus forming an equivalent fraction.

Note that sometimes we have to simplify a ratio in order to find the equivalent ratio.

Example: Find x: $\frac{7}{x} = \frac{21}{33}$

Notice that the numerator on the ratio on the right is 3 times the numerator of the ratio of the fraction on the left. That is, $7 \bullet 3 = 21$. So if we divide both the numerator and the denominator of $\frac{21}{33}$ by 3, we'll form an equivalent ratio (we're just simplifying the ratio!) with a 7 in the numerator, thus finding the value of x.

$$\frac{21}{33} = \frac{21 \div 3}{33 \div 3} = \frac{7}{11}$$

x has to equal 11 in order to form an equivalent ratio with a 7 in the numerator.

Keeping Perspective

Ratios and proportions are very common applications of fractions. Want to describe out how much you're making *per* hour if you make \$8.50 per hour? Write a ratio: $\frac{\$8.50}{\text{hr}}$. Want to make a triple batch of a solution to remove wallpaper? Set up a proportion! Remember, math is a useful tool.

1.6 Rates

As we use math outside of a textbook, we end up needing to measure different aspects of the world, such as time, distance, weight, etc. To do this, we use units of measure — agreed upon periods of time, distance, weight, etc., we can use to compare with other units.

For example, we use a second to represent a certain period of time . . . and 60 seconds to represent a minute . . . and 60 minutes to represent an hour.

In this lesson, we're going to take a look at how to work with ratios of units of measure. The great news is that we can treat units of measure just like we would unknowns! So you'll find we'll be applying the same principles we looked at in the last couple of lessons.

But be careful. One of the biggest challenges of working with units of measure in algebra is remembering that **multi-letter units need to be treated as a single entity**. For example, min does *not* mean m times i times n . . . instead, it means minutes.

Understanding and Simplifying Rates

Since we can work with units just like unknowns, we can end up with some interesting units! Speed equals distance divided by time, or $s = \frac{d}{t}$. If your distance is in meters (m) and your time in seconds (s), then when you divide the two, you get $\frac{\text{m}}{\text{s}}$ (i.e., meters *per* second).

Notice that we simplified $\frac{10\text{ m}}{2\text{ s}}$ just like we did other fractions!

Example: A robot travels a distance (d) of 10 meters in 2 seconds (t). What is its speed (s)?

$$s = \frac{d}{t}$$

$$s = \frac{10\text{ m}}{2\text{ s}} = \frac{5\text{ m}}{\text{s}} = 5\,\frac{\text{m}}{\text{s}}$$

Notice that we wrote the 5 in $5\,\frac{\text{m}}{\text{s}}$ in front of the $\frac{\text{m}}{\text{s}}$. We could have also written it in the numerator, as $\frac{5\text{ m}}{\text{s}}$.

Both $5\,\frac{\text{m}}{\text{s}}$ and $\frac{5\text{ m}}{\text{s}}$ mean the *same thing*. Think about it. View the units of measures like you would unknowns. $5\,\frac{\text{m}}{\text{s}}$ then means 5 *times* $\frac{\text{m}}{\text{s}}$. How would we complete that multiplication? We'd multiply 5 by the numerator, giving us $\frac{5\text{ m}}{\text{s}}$.

Notice also that we were dealing with a *unit* that was a ratio, or **rate**. (A rate is a specific type of ratio. The exact definition for what makes a ratio a rate varies, but the important thing is to know that all rates are ratios — comparisons of 2 quantities using division.)[16] You're used to working with units that are ratios whenever you see speed limits — they give the speed in miles *per* hour, or $\frac{\text{mi}}{\text{hr}}$. In other words, you're looking at how many miles

With ratios or rates, you can read the fraction line as *per*. For example, you can read $\frac{\text{m}}{\text{s}}$ as "meters per second."

you can go in one hour. Likewise, $\frac{\text{m}}{\text{s}}$ means meters *per* second, or how many meters you can go in a second.

Converting Rates

You'll sometimes need to convert rates into other units. For example, say you're driving through Canada and are told the speed limit on a highway is $80\,\frac{\text{km}}{\text{hr}}$. You want to know how many $\frac{\text{mi}}{\text{hr}}$ that is.

Here's the math:

$$80\,\frac{\text{km}}{\text{hr}}\left(\frac{1\text{ mi}}{1.609344\text{ km}}\right) = 80\,\frac{\cancel{\text{km}}}{\text{hr}}\left(\frac{1\text{ mi}}{1.609344\ \cancel{\text{km}}}\right) = \frac{80\ \text{ mi}}{1.609344\text{ hr}} \approx 49.710\,\frac{\text{mi}}{\text{hr}}$$

It's worth noting that, while speed limits are written to show how far you can go in one hour at that speed, they can also be written over different units of time. In fact, in measuring things in science we'll often come up with different rates, such as the one we encountered in an earlier example:

$$\frac{10\text{ m}}{2\text{ s}}$$

This is a speed of 10 meters *per* 2 seconds.

Notice that we wrote the conversion ratio so that the km was in the denominator so it would cancel out with the km in the numerator. We simplified, just as we did with unknowns. Remember, we treat units of measure just as we would unknowns.

Always arrange your conversion ratio so the unit you want to replace will cancel out, leaving your answer in the desired unit.

Now, what if you want to change both the units in a rate? You'll need to multiply it by more than 1 conversion ratio!

Example: Convert $80\frac{\text{km}}{\text{hr}}$ to $\frac{\text{mi}}{s}$.

$$80\frac{\cancel{\text{km}}}{\cancel{\text{hr}}}\left(\frac{1\text{ mi}}{1.609344\ \cancel{\text{km}}}\right)\left(\frac{1\ \cancel{\text{hr}}}{60\ \cancel{\text{min}}}\right)\left(\frac{1\ \cancel{\text{min}}}{60\ s}\right) = \frac{80\text{ mi}}{(1.609344)(60)(60)\ s}$$

$$\approx 0.014\frac{\text{mi}}{s}$$

More with Converting Rates

Let's look at another conversion example. Notice that we rewrite the number as part of the numerator to avoid forgetting to multiply it.

Example: Convert $2\,\frac{\text{m}}{\text{s}}$ to $\frac{\text{m}}{\text{min}}$.

$$\frac{2\text{ m}}{\cancel{\text{s}}}\left(\frac{60\ \cancel{\text{s}}}{1\text{ min}}\right) = \frac{120\text{ m}}{\text{min}} = 120\frac{\text{m}}{\text{min}}$$

Multiplying by a Rate

We can also end up multiplying units together. For example, force is a measure of mass (kg) times the acceleration $\left(\frac{\text{m}}{\text{s}^2}\right)$. So its unit is kg • $\frac{\text{m}}{\text{s}^2}$, which can also be written as $\frac{\text{kg} \bullet \text{m}}{\text{s}^2}$.

If you recall from previous courses, s^2 means s • s. We'll review exponents more thoroughly in the next chapter.

Dividing Units of Measure by Rates

Sometimes when solving problems we end up with crazy units of measure that need simplified. For example, in a future lesson we'll work with a problem that will leave us with this fraction:

$$\frac{40 \text{ mi}}{\frac{35 \text{ mi}}{\text{hr}}}$$

Yikes! How do we simplify this?

Again, just remember to view the units like you would unknowns. We saw in Lesson 1.2 that when we have a fraction divided by a fraction, such as $\frac{1}{5} \div \frac{2}{7}$, which we would write as $\frac{\frac{1}{5}}{\frac{2}{7}}$, we complete the division by inverting and multiplying.

$$\frac{\frac{1}{5}}{\frac{2}{7}} = \frac{1}{5}\left(\frac{7}{2}\right) = \frac{1(7)}{5(2)} = \frac{7}{10}$$

We can do that same thing with units of measure! $\frac{40 \text{ mi}}{\frac{35 \text{ mi}}{\text{hr}}}$ really means 40 mi ÷ $\frac{35 \text{ mi}}{\text{hr}}$. **We can complete this division by inverting the denominator and multiplying!**

$$\frac{40 \text{ mi}}{\frac{35 \text{ mi}}{\text{hr}}} = 40 \cancel{\text{mi}}\left(\frac{\text{hr}}{35 \cancel{\text{mi}}}\right) \approx 1.143 \text{ hr}$$

Notice that when we did that, the miles (mi) canceled out, leaving us with an answer in hours. We now know that it would take us 1.143 hr to go 40 mi at a speed of $35\frac{\text{mi}}{\text{hr}}$.

Once again, the key to working with units of measure is to view them like you would unknowns.

Keeping Perspective

Hopefully that helped you feel more comfortable with rates. They prove very useful as we apply math. Even representing speed limits uses a rate (miles *per* hour).

As you review rates and units of measure in general, remember that God knows the measure of all of creation. He is worthy of all our praise.

> *Who hath measured the waters in the hollow of his hand, and meted out heaven with the span, and comprehended the dust of the earth in a measure, and weighed the mountains in scales, and the hills in a balance? (Isaiah 40:12)*

1.7 Adding and Subtracting Fractions

Adding and subtracting fractions is simple if you remember that **the denominators must be the same**. After all, we can only add and subtract divisions by the same quantity!

For example, say you have $\frac{1}{2}$ of a pie plus another $\frac{1}{3}$ of a pie left over from Thanksgiving dinner. What part of a pie altogether do you have? In other words, what is $\frac{1}{2} + \frac{1}{3}$?

Note that we can't just add the numerators, as the divisions are by different amounts. But if we multiply each fraction by a fraction worth 1 that would give a common denominator, then we can add them together. To find a common denominator, just multiply the denominators together. $2 \bullet 3 = 6$, so 6 is a common denominator. Let's multiply both fractions by a fraction worth one to reach that common denominator.

$$\frac{1}{2}\left(\frac{3}{3}\right) + \frac{1}{3}\left(\frac{2}{2}\right) = \frac{3}{6} + \frac{2}{6}$$

Now we can add the numerators, as we're dealing with divisions by the same amount. This would be the same as adding the pieces of the pie together.

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

We have $\frac{5}{6}$ of a pie left.

Adding and Subtracting Rates

Say a bowling ball is traveling at 8 $\frac{\text{m}}{\text{s}}$. If it decreases its speed by 80 $\frac{\text{m}}{\text{min}}$ before it reaches the pins, how fast is it now going?

$$8\frac{\text{m}}{\text{s}} - 80\frac{\text{m}}{\text{min}}$$

Notice that the units are fractions — we're dealing with rates. Remember that we can write the measurements as part of the numerator.

$$\frac{8\text{ m}}{\text{s}} - \frac{80\text{ m}}{\text{min}}$$

Notice that our denominators are different. When adding and subtracting fractions, **the denominators must be the same** (that is, they have to have what we refer to as a **common denominator**). Otherwise you'd be adding and subtracting divisions by different quantities!

Now, we could either solve this problem by converting $\frac{80\text{ m}}{\text{min}}$ to $\frac{\text{m}}{\text{s}}$ or by converting $\frac{8\text{ m}}{\text{s}}$ to $\frac{\text{m}}{\text{min}}$. Since we know that 60 s equals 1 min (see Appendix B: Reference Section for conversion ratios like this one that show us how many of one unit equal another), we'd have these conversions:

$\frac{60\text{ s}}{1\text{ min}}$ is a fraction worth one, as both 60 seconds and 1 minute represent the same amount of time.

Converting to $\frac{\text{m}}{\text{min}}$:

$$\frac{8\text{ m}}{\cancel{\text{s}}}\left(\frac{60\text{ }\cancel{\text{s}}}{1\text{ min}}\right) = \frac{480\text{ m}}{1\text{ min}}$$

Converting to $\frac{\text{m}}{\text{s}}$:

$$\frac{80\text{ m}}{\cancel{\text{min}}}\left(\frac{1\text{ }\cancel{\text{min}}}{60\text{ s}}\right) = \frac{80\text{ m}}{60\text{ s}} \approx \frac{1.333\text{ m}}{\text{s}}$$

Now we can subtract the fractions:

$$\frac{480\text{ m}}{1\text{ min}} - \frac{80\text{ m}}{\text{min}} = \frac{400\text{ m}}{\text{min}} = 400\frac{\text{m}}{\text{min}}$$

$$\frac{8\text{ m}}{\text{s}} - \frac{1.333\text{ m}}{\text{s}} = \frac{6.667\text{ m}}{\text{s}} = 6.667\frac{\text{m}}{\text{s}}$$

Note: We didn't have to write the 400 to the left of $\frac{\text{m}}{\text{min}}$ or 6.667 to the left of $\frac{\text{m}}{\text{s}}$; an answer of $\frac{400\text{ m}}{\text{min}}$ or $\frac{6.667\text{ m}}{\text{s}}$ would be just as correct. It's just that the numerical value to the left of the units is a little easier to read.

Both 400$\frac{\text{m}}{\text{min}}$ and 6.667$\frac{\text{m}}{\text{s}}$ represent the ***same rate*** — they are just using different units. One is measuring how many meters are traveled per *minute*, and the other how many meters are traveled per *second*. **Always remember to keep track of your units.** While you do not necessarily have to write them out in each step, be sure to think through units and include them in your answer!

More with Rates

While we're talking about rates, it's important to backtrack and point out that we can only add and subtract **like units**. So not only do the denominators need to be the same to add, but to really complete the addition in the numerator, you'll need to make sure you have like units there as well.

For example, if one robot went $1\frac{\text{ft}}{\text{s}}$ and another went $2\frac{\text{in}}{\text{s}}$, we have the same denominator so we can add the numerators, giving us $\frac{1\text{ ft} + 2\text{ in}}{\text{s}}$. But we can't add 1 ft and 2 in until we first convert to the same unit of measure.

$$\frac{1\text{ ft} + 2\text{ in}}{\text{s}} = \frac{12\text{ in} + 2\text{ in}}{\text{s}} = \frac{14\text{ in}}{\text{s}} = 14\frac{\text{in}}{\text{s}}$$

We rewrote 1 ft as 12 in, since both 1 ft and 12 in represent the same length.

Example: Add $2\frac{\text{ft}}{\text{s}}$ and $5\frac{\text{in}}{\text{min}}$. Give your answer in $\frac{\text{in}}{\text{min}}$.

Notice that to completely add these and get the answer in $\frac{\text{in}}{\text{min}}$, we need to convert the $2\frac{\text{ft}}{\text{s}}$ to $\frac{\text{in}}{\text{min}}$. How do we convert both units in the numerator *and* those in the denominator? The same way we have been, only we'll have to multiply by 2 separate conversion ratios: one to convert each unit.

Conversion ratio to convert seconds to minutes

$$2\frac{\text{ft}}{\text{s}}\left(\frac{60\text{ s}}{1\text{ min}}\right)\left(\frac{12\text{ in}}{1\text{ ft}}\right) = 2\frac{\cancel{\text{ft}}}{\cancel{\text{s}}}\left(\frac{60\cancel{\text{ s}}}{1\text{ min}}\right)\left(\frac{12\text{ in}}{1\cancel{\text{ ft}}}\right) = \frac{1{,}440\text{ in}}{\text{min}} = 1{,}440\frac{\text{in}}{\text{min}}$$

Conversion ratio to convert ft to in

Notice that we arranged the conversion ratio however needed to get the appropriate units to cancel out, leaving the requested units.

It's worth noting that we could have done this in 2 steps — converting the feet to inches, and then converting the seconds to minutes. But it saves time to do it all in one step.

Now, we can add this to $5\frac{\text{in}}{\text{min}}$ and get a final answer of $1{,}445\frac{\text{in}}{\text{min}}$.

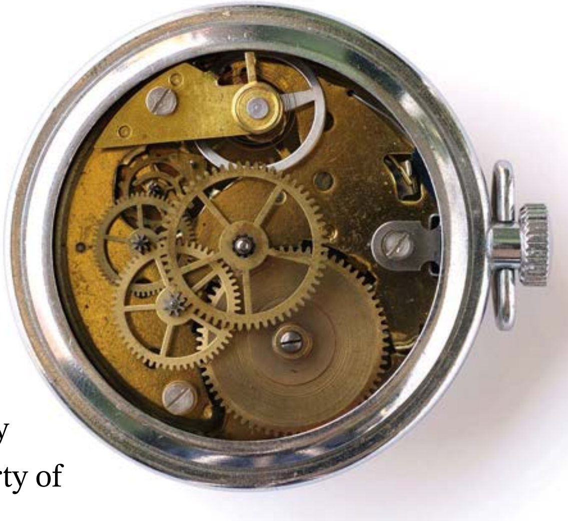
Dealing with Unknowns

Once again, because of the consistent way God governs all things, we can apply what we know about adding fractions together to unknowns as well! As with known values, the **denominators have to be the same, so we're adding or subtracting divisions by the same quantity.**

Example: Simplify $\frac{a}{2} - \frac{1}{b}$.

Example Meaning: The gear ratio of one gear minus the gear ratio of another gear, where the values a and b are not known.

$$\frac{ab}{2b} - \frac{2}{2b} = \frac{ab - 2}{2b}$$



Do you see what we did? In order to get $2b$ as the denominator in the first fraction, we had to multiply it by $\frac{b}{b}$. This was multiplying by a value worth 1, which doesn't change the value (the identity property of multiplication).

$$\frac{a}{2}\left(\frac{b}{b}\right) = \frac{ab}{2b}$$

In order to get $2b$ as the denominator in the second fraction, we had to multiply it by $\frac{2}{2}$, which is also worth 1.

$$\frac{1}{b}\left(\frac{2}{2}\right) = \frac{2}{2b}$$

We then combined them by subtracting the second numerator from the first. Since we can't actually complete the subtraction of ab minus 2 as we don't know the value of ab, we leave it written as a subtraction.

$$\frac{ab}{2b} - \frac{2}{2b} = \frac{ab - 2}{2b}$$

If you're not sure what denominator both fractions can be written as, just multiply the denominators together to find a common denominator to use. Notice that the $2b$ denominator we used above was 2 (the denominator of the first fraction) times b (the denominator of the second fraction).

Example: Simplify $2 + \frac{a}{n}$.

Example Meaning: A $2 base allowance plus whatever additional amount our parents decide to give us divided by the number of siblings they're dividing the additional amount among.

We need to write 2 as a fraction of n. Remember from the last lesson that we can think of 2 as $\frac{2}{1}$.

$$\frac{2}{1} + \frac{a}{n}$$

Notice that we rewrote $1n$ as simply n. We were applying the identity property of multiplication again, knowing that any number times 1 equals itself.

We'll multiply the first fraction by $\frac{n}{n}$, a fraction worth 1 (provided n doesn't equal 0), to get a common denominator. We'll then be able to add the numerators.

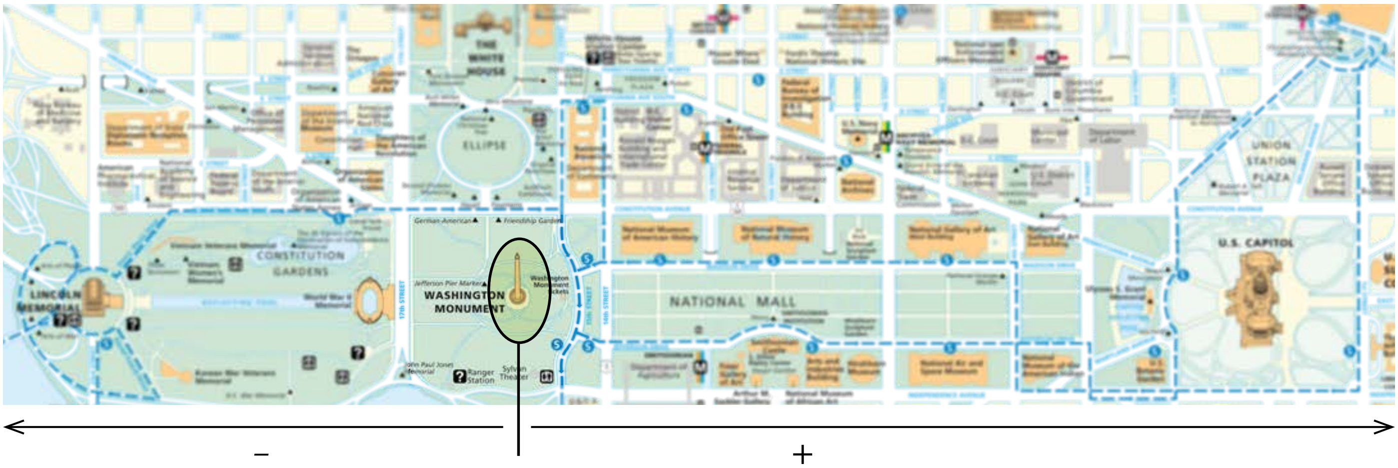
$$\frac{2}{1}\left(\frac{n}{n}\right) + \frac{a}{n} = \frac{2n}{1n} + \frac{a}{n} = \frac{2n}{n} + \frac{a}{n} = \frac{2n + a}{n}$$

Keeping Perspective

Hopefully a lot of what we've been exploring with fractions is review for you. If not, take some extra time to make sure you understand fractions and units of measure, as we use them extensively when we apply math to help us describe God's creation.

1.8 Negative Numbers

As we apply math, we encounter quantities that are the *opposite* of other quantities. We call these **negative numbers**. For example, if you *owe* \$3, you have –\$3, which is *the opposite of* having \$3. Or if you travel in the *opposite* direction of some landmark, you could say you were traveling in the *negative* direction.

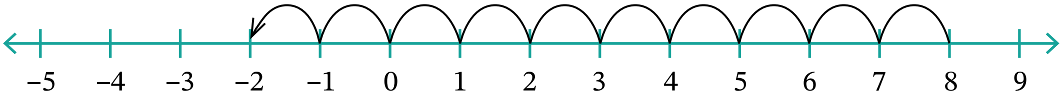


Adding and Subtracting with Negative Numbers

It can be helpful to picture a number line when adding and subtracting negative numbers.

Example: Find 8 – 10.

If we start at 8, and go 10 spaces to the left, we'll get to 0 after 8 spaces. But we still have 2 to go! Thus our answer will be –2.

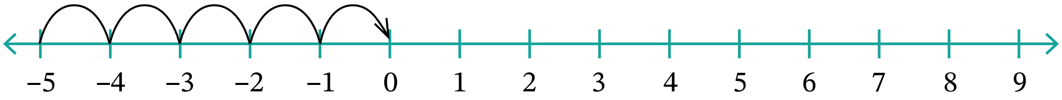


Note that you can view subtraction as an addition of a negative number.

Example: $(5 - x) = (5 + -x)$

It's also worth noting that when you add a number to its opposite, you end up with 0.

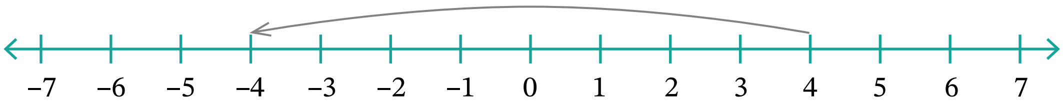
Example: $-5 + 5 = 0$



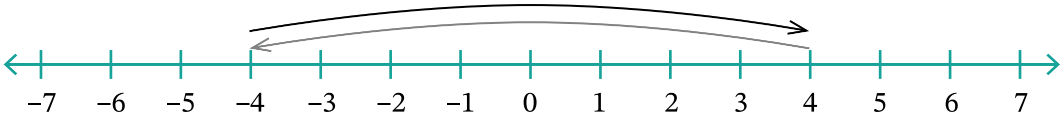
Example: $-a + a = 0$

Multiple Negative Signs

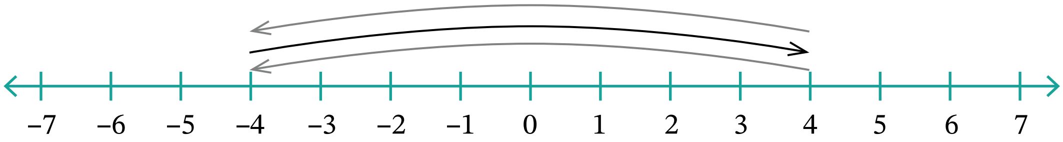
View each negative sign in front of a number as *the opposite of*.



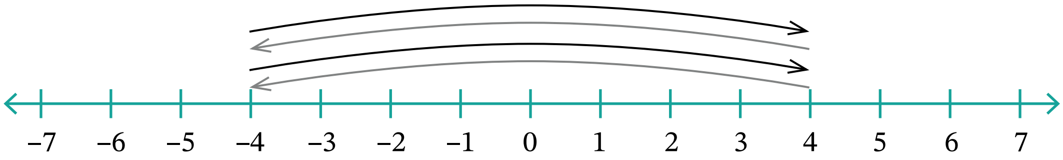
$$-4 = \textit{the opposite of } 4 = -4$$



$$-(-4) = \textit{the opposite of the opposite of } 4 = 4$$



$$-(-(-4)) = \textit{the opposite of the opposite of the opposite of } 4 = -4$$



$$-(-(-(-4))) = \textit{the opposite of the opposite of the opposite of the opposite of } 4 = 4$$

Two negative signs yield a positive answer, three a negative, etc. In fact, you can easily figure out whether the final number is a negative or a positive by saying "negative, positive, negative, positive . . . " as you read off the negative signs. If you end with a negative, the answer is negative; if you end with a positive, the answer is positive. **In short, if there are an odd number of negative signs, the answer will be negative; an even number, the answer will be positive.**

Odd numbers cannot perfectly divide by 2, while even numbers can. See Lesson 3.2 for a more precise definition.

Multiplying and Dividing Negative Numbers

The negative, positive, negative, positive, etc., rule (i.e., odd number of negative signs means negative answer, and even means positive answer) holds true both with negative signs in front of a value *and* when multiplying or dividing.

In fact, one very helpful way of thinking about negative signs in front of a number or unknown is as a multiplication by –1. After all, we can multiply any number by 1 without changing the value. Thus $-a$ equals $-1a$. In fact, many find it useful to think of $-a$ as –1 times a, or $(-1)a$. Viewing it as a multiplication by –1 can help in determining if the negative signs cancel out or not. Remember that each –1 takes the *opposite*, making it negative, positive, etc.

Example: Simplify $-(-(-a))$.

$$-(-(-a)) = (-1)(-1)(-1)a = (-1)a = -a$$

The opposite of the opposite of the opposite of a is $-a$.

Read negative, positive, negative.

It's not necessary to write out all the −1s . . . they're included just in case that's a helpful way for you to think of the negative signs.

Odd number of signs → negative answer

Example: Find $-a$ if a is -3.

Substitute -3 for a: $-(-3) = (-1)(-1)3 = 3$

The opposite of the opposite of 3 is +3!

Even number of signs → positive answer

Example: Simplify $\frac{-4}{-b}$.

$$\frac{-4}{-b} = \frac{(-1)(4)}{(-1)(b)} = \frac{4}{b}$$

Here we have a negative number divided by another negative number. So, our answer ends up with *two* negative signs . . . *the opposite of the opposite* . . . which will be positive.

Even number of signs → positive answer

Another way of thinking about this is that the negative signs cancel out, as one is in the numerator and the other in the denominator.

$$\frac{-4}{-b} = \frac{(\cancel{-1})(4)}{(\cancel{-1})(b)} = \frac{4}{b}$$

Example: Simplify $-\frac{4}{-b}$.

$$-\frac{4}{-b} = (-1)\frac{4}{(-1)(b)} = \frac{4}{b}$$

Even number of signs → positive answer

Another way of thinking about this is that we have a positive number (4) divided by a negative number ($-b$). So, the $\frac{4}{-b}$ part will be negative . . . but then the negative sign in front tells us to take the opposite of that, which would make the final answer positive.

You can also think of there being a negative sign in the numerator and one in the denominator, as $(-1)\frac{4}{(-1)(b)}$ can be rewritten as $\frac{(-1)4}{(-1)(b)}$. The –1s would then cancel out.

Example: Simplify $-\frac{-4}{-b}$.

$$-\frac{-4}{-b} = -1\left(\frac{(\cancel{-1})(4)}{(\cancel{-1})(b)}\right) = -1\left(\frac{4}{b}\right) = -\frac{4}{b}$$

Notice that we had a total of 3 negative signs. Thus, we had *the opposite of the opposite of the opposite*, or a negative answer.

Odd number of signs → negative answer

Another way of thinking about this is that –4 divided by –*b* will be positive, and then the negative sign in front will make that negative again.

Example: Simplify $\frac{3x}{2} - \frac{1}{-2}$.

$$\frac{3x}{2} - \frac{1}{-2} = \frac{3x}{2} + \frac{1}{2} = \frac{3x+1}{2}$$

We could have rewritten $-\frac{1}{-2}$ as $(-1)\left(\frac{1}{(-1)2}\right)$. Notice that then we'd have a –1 in the numerator and the denominator . . . which would cancel out. Thus we end up with positive $\frac{1}{2}$ to add to $\frac{3x}{2}$.

Keeping Perspective

Negative numbers give us a way of describing that a quantity is *the opposite* of something else. We can use them to represent going in *the opposite direction*, owing money (which is *the opposite of* having it — examples would be money you owe on your rent or mortgage), or current flowing in *the opposite direction*. Make sure you're comfortable with working with negative numbers, as they'll come up both throughout this course and in real life.

1.9 Chapter Synopsis

Well, we've reached the end of our first chapter together. In this chapter, we reviewed core concepts that we will use again and again and again and again (did we say that enough times?) as we dig into algebra. All of these concepts rest on the consistent way God governs all things. Our very ability to rely on multiplication to always work the same way, for example, reminds us that God is faithful day after day to govern all things consistently. And He'll be just as faithful to everything else He's said in His Word!

At the end of each chapter, there's typically a review day. But since this chapter was a review chapter, there's no review day. Take just a minute, though, to look over the key skills from this chapter to make sure you're ready for the quiz.

Key Skills for Chapter 1

Understand and apply key properties of addition and multiplication. (Lesson 1.1)

- **Commutative** (order doesn't matter):

  $a + b = b + a$ (for addition)

  $ab = ba$ (for multiplication)

- **Associative** (grouping doesn't matter):

  $(a + b) + c = a + (b + c)$ (for addition)

  $(ab)c = a(bc)$ (for multiplication)

- **Identity** (can multiply by 1 or add 0 without changing value):

  $a + 0 = a$ (for addition)

  $1a = a$ (for multiplication)

Know that any number (except 0) divided by itself equals 1 and that we cannot divide by 0. (Lesson 1.1)

Examples: $\frac{a}{a} = 1$, provided $a \neq 0$

In $\frac{x}{a}$, a cannot equal 0.

Know how to insert known values into a formula. (Lesson 1.1)

Understand the concept of equality and inequality. (Lesson 1.2)

Know various terms (including expressions, equations, and simplify) and conventions (such as that $5x$ means 5 times x and that capital and lowercase letters cannot be used interchangeably — V is a different unknown than v). (Lesson 1.2)

Understand fractions and how to work with them. (See chart; Lessons 1.2–1.4, 1.7)

Understand ratios and proportions and how to work with them. (Lesson 1.5)

- Ratios are comparisons via division; we can write them as a fraction and work with them like fractions.
- A proportion is 2 equal ratios; you can find missing values in a proportion by figuring out what the value would have to be to form an equivalent fraction.

  Example: $\frac{7}{x} = \frac{21}{33}$; $x = 11$

  (The first fraction has to be $\frac{7}{11}$ to form an equivalent fraction with 7 in the numerator — if we multiplied both the numerator and denominator by $\frac{3}{3}$, we'd get $\frac{21}{33}$.)

Understand how to work with units of measure and rates. (Lesson 1.2, Lesson 1.6)

- Multiplying with units (make sure you maintain the units).

  Example: $(7 \text{ kg})(3 \text{ m}) = 21 \text{ kg} \bullet \text{m}$
- Treat units of measure like you would unknowns, only viewing the *entire unit* as a single unknown. For example, treat min (the abbreviation for *minute*) as a single value, not as *m* times *i* times *n*.
- Converting units (multiply by a conversion ratio worth 1) and simplifying units of measure.

  Example: $$80\frac{\text{km}}{\text{hr}}\left(\frac{1 \text{ mi}}{1.609344 \text{ km}}\right) = 80\frac{\cancel{\text{km}}}{\text{hr}}\left(\frac{1 \text{ mi}}{1.609344 \cancel{\text{km}}}\right) = \frac{80 \text{ mi}}{1.609344 \text{ hr}}$$
  $$\approx 49.71\frac{\text{mi}}{\text{hr}}$$
- You can only add and subtract like units.

  Example: $\frac{1 \text{ ft} + 2 \text{ in}}{\text{s}} = \frac{12 \text{ in} + 2 \text{ in}}{\text{s}} = \frac{14 \text{ in}}{\text{s}} = 14\frac{\text{in}}{\text{s}}$

Understand negative numbers and how to work with them. (Lesson 1.8)

- **Each negative sign means "the opposite of."**

  Odd number of negative signs = negative answer

  Even number of negative signs = positive answer

- **Addition and Subtraction**

  Add and subtract by thinking about how many times it takes to get to 0 and then how much beyond 0 you have to go.

  Examples: $3 - 4 = -1$

  $$x - (-x) = x + x = 2x$$

  You can view subtraction as an addition of a negative number.

  Example: $(5 - x) = (5 + -x)$

  When you add a number to its opposite, you end up at 0.

  Example: $-a + a = 0$

- **Multiplication and Division**

  When multiplying and dividing or looking at multiple negative signs in front of an expression, an even number of negative signs gives a positive result, and an odd number of negative signs gives a negative result. You can also read off negative, positive, negative, etc.

  Examples: $(-1)(-x) = x$

  $$-(6 - x) = -6 + x$$

  $$\frac{-7x}{2} = -\frac{7x}{2}$$

  $$-\frac{7x}{-2} = \frac{7x}{2}$$

  $$-\frac{-7x}{-2} = -\frac{7x}{2}$$

Fractions

Fractions represent division.

- **Addition**

  Denominators must be the same; then add numerators.

  Example: $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

- **Subtraction**

  Denominators must be the same; then subtract the second numerator from the first.

  Example: $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$

- **Multiplication**

  Multiply numerators and denominators.

  Example: $\frac{a}{b}\left(\frac{c}{d}\right) = \frac{ac}{bd}$

  Treat non-fractional quantities as if they had a 1 in the denominator, as dividing by 1 doesn't change the value.

  Example: $x\left(\frac{a}{b}\right) = \frac{x}{1}\left(\frac{a}{b}\right) = \frac{xa}{1b} = \frac{xa}{b}$

- **Division**

  Invert number being divided by and multiply.

  Example: $\frac{\frac{a}{b}}{\frac{c}{b}} = \frac{a}{b}\left(\frac{b}{c}\right) = \frac{ab}{bc} = \frac{a}{c}$

- **Reciprocal or Inverse**

  The number that times it equals 1. Flip the numerator and denominator to find.

  Examples: The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

  The reciprocal of 4 is $\frac{1}{4}$.

Chapter 1 Endnotes:

1 Walter W. Sawyer, *Mathematician's Delight* (Harmondsworth Middlesex: Penguin, 1943), p. 10, quoted in James D. Nickel, rev. ed., *Mathematics: Is God Silent?* (Vallecito, CA: Ross House Books, 2001), p. 290.

2 *New Oxford American Dictionary*, 3rd edition (Oxford University Press, 2012), Version 2.2.1 (156) (Apple, 2011), s.v., "algebra," quoted in Loop, *Principles of Mathematics: Book 2*, p. 176.

3 Leonard Euler, *Elements of Algebra, by Leonard Euler, Translated from the French; with the Additions of La Grange, and the Notes of the French Translator* (London: J. Johnson and Co., 1810), https://books.google.com/books?id=hqI-AAAAYAAJ&pg=PR1#v=onepage&q&f=false, p. 270; cited in Katherine A. Loop, *Principles of Mathematics: Book 2* (Green Forest, AR: 2016), p. 176.

4 *New Oxford American Dictionary*, s.v., "formula."

5 When rounding, look at the value to the right of the place to which you want to round. If it is 5 or greater, round up; if it is less than 5, you just round down, or in the case of decimals, leave the place you're rounding to as it is. For example, if rounding 9.578542 to the 3rd decimal, we would look at the 4th decimal place, which is a 5. Since 5 is 5 or greater, we'd round the 3rd decimal place up to the next number, giving us a rounded value of 9.579. But if we had 9.578342 instead, we'd round to 9.578 instead.

6 See Appendix B: Math's Message.

7 Lexico.com, s.v. "convention," https://www.lexico.com/en/definition/convention.

8 Top Row: One of several possible symbols the Egyptians may have used as a form of an equals sign (this particular hieroglyphic means "together" and could have been used to symbolize the results of addition); symbol used by Diophantus (200s); form of modern symbol presented by Recorde (1557); symbol used by Buteo (1559); symbol used by Holzman, better known as Xylander, that several other mathematicians adopted.

9 See Dictionary.com Unabridged, based on *Random House Unabridged Dictionary*, s.v. "kilogram-meter," https://www.dictionary.com/browse/kilogram-meter.

10 *New Oxford American Dictionary*, s.v., "expression."

11 Ibid. s.v., "equation."

12 Florian Cajori, *A History of Mathematics*, 5th rev. ed. (NY: Chelsea Publishing, 1991), p. 123.

13 Lexico.com, s.v. "division," https://www.lexico.com/definition/division.

14 *The American Heritage Dictionary of the English Language*, 1980 New College Edition, s.v. "ratio."

15 See PLTW, Inc., *Engineering Formulas* (n.p., n.d.), p. 6., and Woodgears.ca, "Gear Ratios and Compound Gear Ratios," (n.d.), https://woodgears.ca/gear/ratio.html.

16 See http://mathforum.org/library/drmath/view/58042.html for an exploration of some of the different definitions for "rate" verses "ratio." The author concludes with this: "A rate generally involves a 'something else,' either two different kinds of units (such as distance per time), or just two distinct things measured with the same unit (such as interest money per loaned money)." Doctor Peterson, The Math Forum, "Rate vs. Ratio" The Math Forum @ Drexel, http://mathforum.org/library/drmath/view/58042.html, accessed 10/1/14.

Chapter 4

Solving for Unknowns and Problem-Solving Skills

4.1 Solving for Unknowns

While **algebra** can be defined as simply using letters to "represent numbers and quantities,"[1] another definition for algebra encompasses the reason we use those letters: "the science which teaches how to determine unknown quantities by means of those that are known."[2]

You see, because of how unerringly consistent God created and sustains all things, we can work with unknown quantities (represented with letters and symbols) with confidence, knowing that whatever quantities those letters and symbols represent, they will operate according to the "fixed order" God established. And we can use this knowledge to find missing information in real-life problems.

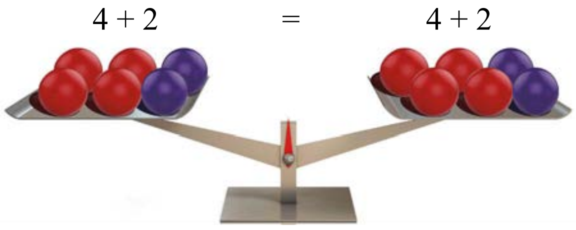
At the core of finding missing information is the concept of equality. **If we know things are equal, we can use that knowledge to find missing information.** You should already be very familiar with the basics of solving for unknowns, but let's briefly review and then explore how to handle equations requiring a few extra steps.

> "Thus says the Lord: If I have not established my covenant with day and night and the fixed order of heaven and earth, then I will reject the offspring of Jacob and David my servant and will not choose one of his offspring to rule over the offspring of Abraham, Isaac, and Jacob. For I will restore their fortunes and will have mercy on them." (Jeremiah 33:25–26; ESV)

Using Equality to Find Unknowns

An equal sign means that two expressions are equal. Now if they're equal, it makes sense that if we add or subtract the same quantity to *both sides* of an equation, the equation will stay in balance.

Added 2 to both sides.



Subtracted 2 from both sides.

Likewise, we can multiply or divide *both sides* of an equation by the same quantity and the sides again stay equal.

Multiplied both sides by 2.

$$4 \cdot 2 \quad = \quad 4 \cdot 2$$

Divided both sides by 2.

$$4 \div 2 \quad = \quad 4 \div 2$$

We can also square both sides of the equation . . . or find the square root of both sides.

Squared both sides.

$$4^2 \quad = \quad 4^2$$

Took the square root of both sides.

$$\sqrt{4} \quad = \quad \sqrt{4}$$

We can perform the *same operation* using the *same amount* to *both sides* of an equation without changing the value.

We can use this knowledge to isolate an unknown on a side by itself, thus finding what it equals.

Example: Solve $5x = 10$ for x.

Now, you can probably tell right away that the answer should be 2, since 5 times 2 equals 10. But let's review how to find that systematically for problems that are less obvious.

Remember, the fraction line means division.

We know what 5 *times* x equals. So, we need to *divide* (the opposite operation of multiplication) both sides by 5 to find x.

$$\frac{5x}{\mathbf{5}} = \frac{10}{\mathbf{5}}$$

Notice that we divided *both sides* by the *same amount*, so the equation is still in balance. Now let's complete the division!

$$x = 2$$

Remember back from Lesson 1.2 that division can be completed by inverting and multiplying — that is, by multiplying by the reciprocal. Notice that we could have multiplied *both sides* of the equation by the reciprocal of 5 — that is, $\frac{1}{5}$ — to reach the same answer.

$$\left(\frac{1}{5}\right)5x = 10\left(\frac{1}{5}\right)$$

$$\frac{1}{\cancel{5}}(\cancel{5}x) = \overset{2}{\cancel{10}}\left(\frac{1}{\cancel{5}}\right)$$

$$x = 2$$

We reach the *same answer* either way, but multiplying by the reciprocal is usually the easier way of dividing both sides, especially when fractions are involved.

You can check to see if you solved a problem correctly by plugging the answer you got back into the original equation and seeing if it holds true. For example, for $5x = 10$, we got an answer of $x = 2$. Let's plug 2 in for x in the original equation and see if it holds true:

$$5x = 10 \qquad \text{(original equation)}$$

$$5(\mathbf{2}) = 10 \qquad \text{(plugged in 2)}$$

$$\mathbf{10} = 10 \qquad \text{(simplified)}$$

Notice that we ended up with equal amounts on both sides of the equal sign. We've now verified that we found the correct value for x.

To check if you found the correct value for an unknown, plug the value you found back into the original equation for the unknown and simplify. If you end up with equal amounts on both sides of the equal sign, you know you found the correct value; if not, try again!

While the correct value might seem obvious in $5x = 10$, knowing how to check your work becomes very important as problems get more complicated. It can help you avoid a lot of errors! And if you were to apply math outside of a textbook and really needed to make sure you had the correct answer in a situation where there wasn't a solution manual, knowing how to check your work comes in really handy! For example, if you were ordering tile and had figured out how much tile you needed, an error could mean you pay for more tile than you really need . . . or could mean you don't have enough to finish the project. Checking your work can sometimes save money and time!

Problems Requiring More Than One Step

Sometimes, problems will require more than one step in order to solve. Don't be surprised if it takes more than one step to isolate the unknown on a side by itself! Just keep working step-by-step until you have the unknown by itself.

Example: Solve $5 = \frac{30}{x}$ for x.

Notice that in this problem, we have x in the denominator. What do we do? Well, let's start by getting it into the numerator. We can do that by multiplying both sides by x.

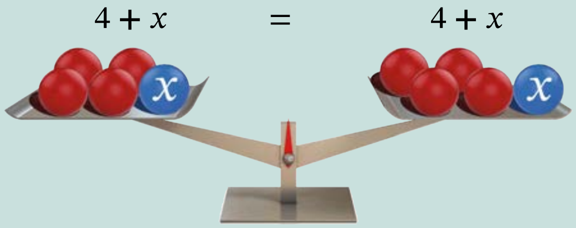
$$5\boldsymbol{x} = \frac{30}{x}\boldsymbol{x}$$

$$5x = \frac{30}{\cancel{x}}\cancel{x}$$

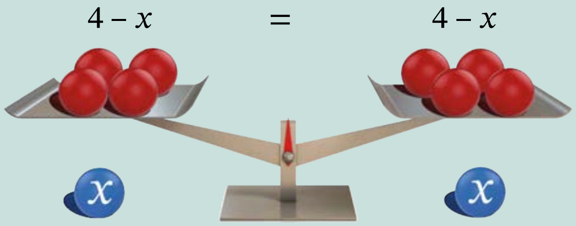
$$5x = \mathbf{30}$$

Note that because of the consistent way God governs all things, it doesn't matter if we know the quantity that we are working with. So long as we know that we're using the same quantity and same operation on *both sides*, we know that the equation will stay in balance.

Added x to both sides.



Subtracted x from both sides.



Now we can divide both sides by 5 or multiply them both by $\frac{1}{5}$ to find x.

$$\frac{5x}{5} \quad \textit{or} \;\; 5x\left(\frac{1}{5}\right)$$

$$\frac{\cancel{5}x}{\mathbf{5}} = \frac{30}{\mathbf{5}} \;\; \textit{or} \;\; \cancel{5}x\left(\frac{\mathbf{1}}{\mathbf{5}}\right) = 30\left(\frac{\mathbf{1}}{\mathbf{5}}\right)$$

$$\boldsymbol{x = 6}$$

Example: Solve $80 = -x + 3$ for x.

We'll start by subtracting 3 from *both sides*.

$$80 \mathbf{- 3} = -x + 3 \mathbf{- 3}$$

$$\mathbf{77 = -}\boldsymbol{x}$$

Remember from Lesson 1.6 that we can think of $-x$ as $(-1)(x)$. So $-x(-1)$ would equal $(-1)(x)(-1)$, which would leave us with positive x.

Now we've found *the opposite of* x, but we want to find x! So we have to multiply *both sides* by –1.

$$77(\mathbf{-1}) = -x(\mathbf{-1})$$

$$\mathbf{-77 =} \boldsymbol{x}$$

$x = -77$ (swapped sides of the equation to get the unknown on the left, as that is easier to read)

Note that equal means equal. $-77 = x$ and $x = -77$ mean the same thing. Only we typically put the unknown on the left to make it easier to read.

Dealing with Unknowns

Sometimes, we want to rewrite equations so as to express the relationship in a different way. For example, if you know that $P = VI$ and you needed to find the voltage (V) when you know the power (P) and current (I) for a bunch of different devices, you would want to first solve the equation for voltage so it would be easier to plug in all the values. And you can do this because of the consistent way God governs all things! It doesn't matter that we don't know the exact quantities we're dealing with; we know that if we perform the same operation using the same quantity to both sides of an equation, that equation will stay in balance.

$$P = VI$$

$$\frac{P}{I} = \frac{VI}{I} \qquad \text{(divided both sides by } I\text{)}$$

$$\frac{P}{I} = V \qquad \text{(simplified)}$$

$$V = \frac{P}{I} \qquad \text{(swapped sides for clarity)}$$

It's important to note that **in dividing both sides of an equation by an unknown, we're assuming the unknown is not 0**. After all, we can't divide by 0. In this case, I can't equal 0.

Applying the Skill

Let's look at some examples of actually solving for real-life unknowns.

Example: If we know that we need to travel 40 miles and that the speed limit on the road is 35 miles per hour, how long will it take us?

Distance equals speed times time. Here we want to figure out the time.

$$distance = speed(time)$$

$$d = st$$

While you might be tempted to now plug in the information we've been given and simplify, there's an easier way.

When solving real-life problems, you'll save yourself some time if **you first rearrange the equation so the unknown you're trying to find is on a side by itself and *then* plug in all the known values.** That way, you're not having to keep track of all the numbers and units while you rearrange the equation.

So before we plug in the information we've been given, let's solve this problem so that t, the value we're trying to find, is on a side by itself.

$$\frac{d}{s} = t \qquad \text{(divided both sides by } s\text{)}$$

And now we can plug in the values we've been given and solve.

$$\frac{40\text{mi}}{\frac{35\text{ mi}}{\text{hr}}} = t \qquad \text{(substituted values given)}$$

$$40\text{ mi}\left(\frac{\text{hr}}{35\text{ mi}}\right) = t \qquad \text{(inverted and multiplied to complete the division — see Lesson 1.4)}$$

$$1.143\text{ hr} \approx t \qquad \text{(simplified)}$$

$$t \approx 1.143\text{ hr} \qquad \text{(swapped sides of the equation)}$$

We can swap the entire sides of an equation without changing the meaning, as we did in the last step above. After all, equal means equal. Note that we used the approximately equal sign at the end to signify that we used a rounded value.

Keeping Perspective

Because of the consistent way God governs all things, if we perform the same operation using the same quantity to both sides of an equation, that equation will stay in balance. We can use this knowledge to help us solve real-life problems.

Take careful note to what we're really doing: we're observing consistencies of God's creation and using those to help us complete the tasks He's given us to do. Algebra would be completely pointless were it not for Jesus' faithfulness in "upholding all things by the word of His power. . . ." (Hebrews 1:3). Remember to praise the Creator today as you practice solving for unknowns.

4.2 Solving for Unknowns Using Roots

In the last chapter, we looked at roots. Now let's connect that with what we looked at in the last lesson regarding solving for unknowns.

Example: Solve $2x^2 = 98$ for x.

Yikes! What do we do? We've got x^2, but we need to find x.

No worries — remember, a squared number means that number multiplied by itself. So, if we can find the value of x multiplied by itself, or x^2, then we can take the square root of *both sides* to find the value of just x.

We'll start by dividing both sides of the equation by 2.

$$\frac{2x^2}{2} = \frac{98}{2}$$

$$x^2 = 49$$

Now let's take the square root of both sides to find x.

$$\pm\sqrt{x^2} = \pm\sqrt{49}$$

$$\pm x = \pm 7$$

Notice that we put the symbol ± (called the plus or minus sign) in front of the square root symbols. After all, a square root *can* be either positive or negative (for example, both $-7(-7)$ and $7(7)$ equal 49). While we define $\sqrt{\ }$ and fractional exponents as meaning the positive root for even roots (i.e., roots where there are 2 possibilities), when writing those signs ourselves to find an unknown, we can't make that assumption!

However, when taking the square root of both sides like this, **it's only necessary to list the ± on one side.** So our answer is as follows:

$$x = \pm 7$$

Why? Well, let's make a list of all of the possibilities we could get from $\pm x = \pm 7$. x could be positive while 7 is positive, or positive while 7 is negative. Or x could be negative while 7 is either positive or negative.

Now let's solve the last 2 equations for x by multiplying both sides by –1.

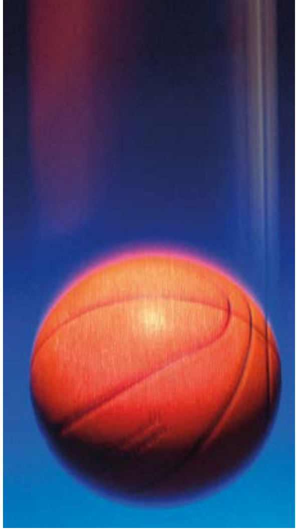
$$-x = 7$$

$x = -7$ (multiplied both sides by –1)

$$-x = -7$$

$x = 7$ (multiplied both sides by –1)

When taking an even root of both sides of an equation, put a ± sign in front of one side of the equation.



In this problem, x could be standing for the time it takes a ball to drop 240.1 m if the only force on it is gravity.

$x = 7$
$x = -7$
$-x = 7$
$-x = -7$

Notice that both of these values (–7 and 7) were already accounted for by the ± sign in front of 7 — basically, x can either be ± 7. Thus, it's not necessary to list the ± sign in front of the x side.

Note that rather than thinking of taking the square root of both sides in the example we just worked through, we could also have thought about the operation in terms of raising both sides to the $\frac{1}{2}$. Again, though, since we're using this to find an unknown and not just solving a fractional exponent already written down, we have to account for the fact that the answer could be positive *or* negative. We'll do that by **putting a ± sign in front of one side of the equation**. Here is the same problem we just solved, done using fractional exponents instead.

$$\begin{aligned} 2x^2 &= 98 && \\ \frac{2x^2}{2} &= \frac{98}{2} && \text{(divided both sides by 2)} \\ x^2 &= 49 && \text{(simplified)} \\ (x^2)^{\frac{1}{2}} &= \pm(49)^{\frac{1}{2}} && \text{(raised both sides to the } \tfrac{1}{2}\text{)} \\ x^{\frac{2(1)}{2}} &= \pm 7 && \\ x^1 &= \pm 7 && \\ x &= \pm 7 && \text{(simplified)} \end{aligned}$$

Remember, we can perform the same operation using the same quantity to both sides without changing the value. **To solve a problem, just figure out what operation will separate the unknown by itself!**

Other Roots

While we emphasize square roots because they are the more common "tool," the same principle applies with other roots! You can take the cubed root, fourth root, etc., of both sides as well. Just remember, even roots can be either positive or negative, while odd roots of positive numbers will be positive.

Note that we'll learn other methods to solve problems where x is raised to powers greater than 2, as sometimes you can miss answers by taking the root of both sides. For example, in $16x = 4x^3$, if you divide both sides by $4x$, you're assuming x doesn't equal 0… but $x = 0$ is a valid solution!

Example: Solve $x^3 = 40$ for x.

$$\begin{aligned} (x^3)^{\frac{1}{3}} &= 40^{\frac{1}{3}} && \text{(taking the cubed root of both sides, using the fractional exponent notation)} \\ x &\approx 3.420 && \end{aligned}$$

Notice that we did not put a ± sign, as a cubed root is an odd root. (The 3 in the denominator tells us the root — and 3 is an odd number; it cannot be evenly divided by 2.)

Units and Square Roots

If a problem has units of measure, remember **that you can take the square root of units of measure just like you would of unknowns,** remembering to view the *entire unit* as a single entity.

Example: Solve $50\text{ mi}^2 = 2x^2$ for x.

$$\frac{50\text{ mi}^2}{2} = \frac{2x^2}{2} \qquad \text{(divided both sides by 2)}$$

$$25\text{ mi}^2 = x^2 \qquad \text{(simplified)}$$

$$\pm\sqrt{25\text{ mi}^2} = \sqrt{x^2} \qquad \text{(took the square root of both sides)}$$

$$\pm 5\text{ mi} = x \qquad \text{(simplified)}$$

Notice that $\sqrt{\text{mi}^2}$ was mi, just like $\sqrt{x^2}$ was just x.

Applying the Skill

This application example is much more involved, so be patient with yourself in reading it through. (You may even want to work it out yourself.) It's designed to help you understand how to solve real-life problems. When taken step-by-step, you have all of the skills you need!

Example: The consistent way God causes the force of gravity to operate between two objects can be expressed like this: $F = G\frac{m_1 m_2}{r^2}$. Let's say we know that m_1 is 500 kg and m_2 is 5,000 kg. G is known to have a value of $(6.67 \times 10^{-11})\left(\frac{\text{m}^3}{\text{kg} \bullet \text{s}^2}\right)$ and we measure the force (F) to be 100,000 N. How would we find what the distance between the center of the masses (r) is?

Once again, rather than instantly substituting the values, **we're going to solve for the unknown we're trying to find first.**

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{(original equation)}$$

$$Fr^2 = Gm_1 m_2 \qquad \text{(multiplied \textit{both sides} by } r^2\text{)}$$

$$r^2 = \frac{Gm_1 m_2}{F} \qquad \text{(divided \textit{both sides} by F)}$$

$$r = \pm\sqrt{\frac{Gm_1 m_2}{F}} \qquad \text{(took the square root of both sides)}$$

Now, since we know that r is representing the distance between these objects and that those distances would be positive, we can omit the ± sign and just look for the positive root, giving us this:

$$r = \sqrt{\frac{Gm_1 m_2}{F}}$$

Notice that we solved the equation for the unknown we needed to find. Now we can simply plug in the values we know and solve for that unknown!

$$r = \sqrt{\frac{\left(6.67 \times 10^{-11}\right)\left(\frac{\mathrm{m}^3}{\mathrm{kg} \bullet \mathrm{s}^2}\right)500\ \mathrm{kg}\left(5{,}000\ \mathrm{kg}\right)}{100{,}000\ \mathrm{N}}}$$

(substituted values given)

$$r = \sqrt{\frac{\left(\mathbf{1.6675 \times 10^{-9}}\right)\left(\frac{\mathrm{m}^3}{\mathrm{kg} \bullet \mathrm{s}^2}\right)\left(\mathrm{kg}\right)\left(\mathrm{kg}\right)}{\mathrm{N}}}$$

(used a calculator to complete all the multiplication and division of the numbers)

$$r = \sqrt{\frac{\left(1.6675 \times 10^{-9}\right)\left(\frac{\mathbf{m^3 \bullet kg}}{\mathbf{s^2}}\right)}{\mathrm{N}}}$$

(simplified the units in the numerator to a single fraction)

$$r = \sqrt{\frac{\left(1.6675 \times 10^{-9}\right)\left(\frac{\mathrm{m}^3 \bullet \mathrm{kg}}{\mathrm{s}^2}\right)}{\frac{\mathbf{kg \bullet m}}{\mathbf{s^2}}}}$$

(replaced N in the denominator with $\mathrm{kg} \bullet \frac{\mathrm{m}}{\mathrm{s}^2}$ — see Lesson 2.5)

$$r = \sqrt{\left(1.6675 \times 10^{-9}\right)\left(\frac{\mathrm{m}^3 \bullet \mathrm{kg}}{\mathrm{s}^2}\right)\left(\frac{\mathbf{s^2}}{\mathbf{kg \bullet m}}\right)}$$

(completed the division by inverting and multiplying the denominator)

$$r = \sqrt{\left(1.6675 \times 10^{-9}\right)\left(\frac{\cancel{\mathrm{m}}^3\mathrm{m}^2 \bullet \cancel{\mathrm{kg}}}{\cancel{\mathrm{s}}^2}\right)\left(\frac{\cancel{\mathrm{s}}^2}{\cancel{\mathrm{kg}} \bullet \cancel{\mathrm{m}}}\right)}$$

(simplified the units)

$$r = \sqrt{\left(1.6675 \times 10^{-9}\right)\mathrm{m}^2} \approx \mathbf{4.084 \times 10^{-5}\ m}$$

(used a calculator to find the square root)

The distance here is really small, so these two objects have a lot of their mass compacted into a small area. This might be true for what are known as black holes, which are found out in the farthest reaches of space and are so dense that any light that goes into them is not reflected back but just gets added to the mass of the black hole.[3]

Always sanity check your answers to real-life problems by seeing if your answer and units of measure make sense. If you plug the numbers into the calculator incorrectly on the above, you might end up with a number like 4,083,503.398 m. This doesn't make any sense, since we were taking the square root of a much smaller number (if we rewrite 1.6675×10^{-9} so it's not in scientific notation, we'd get 0.0000000016675).

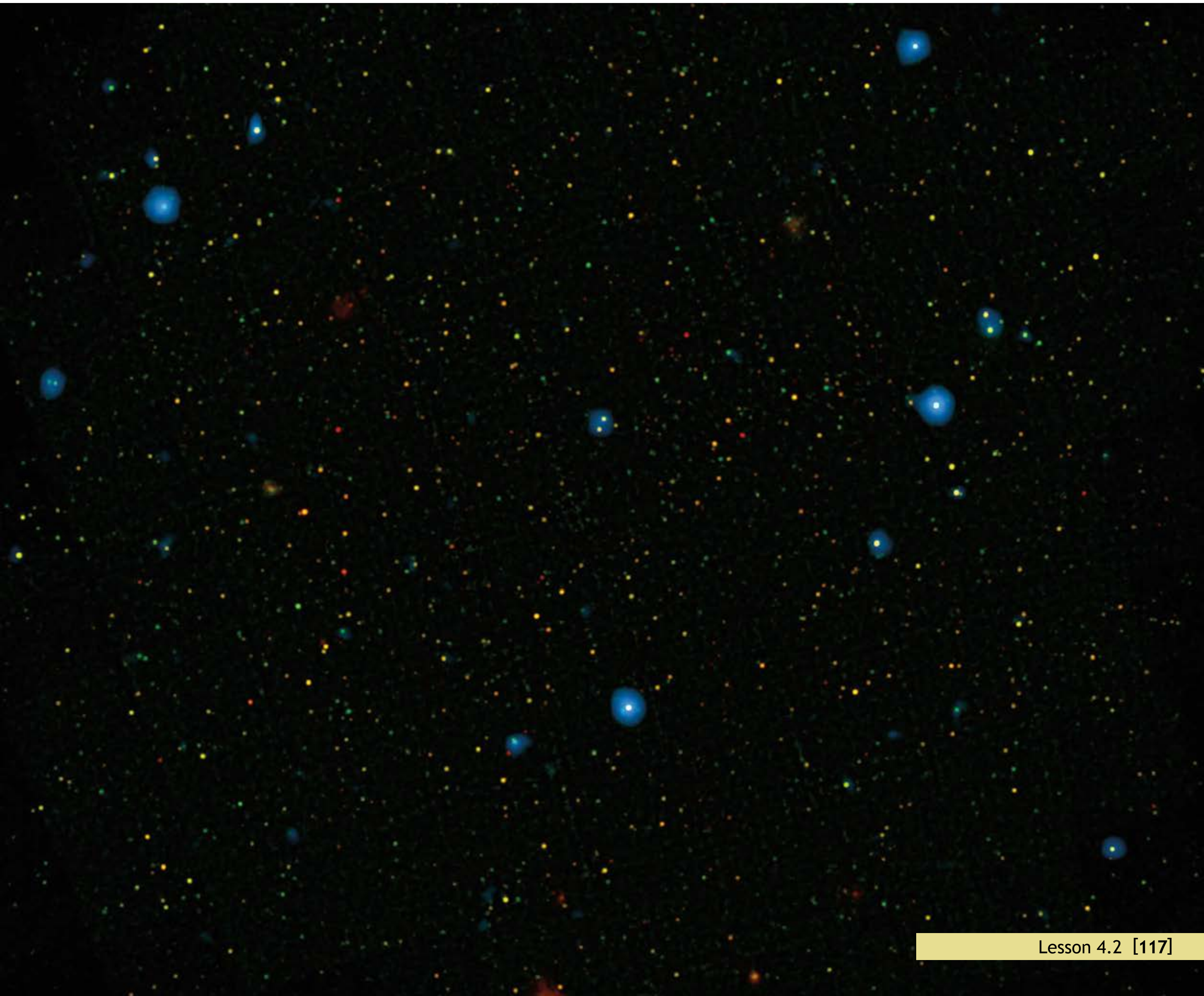
Also, if we ended up with a unit of measure that is not a distance unit of measure, we'd also know that we did something wrong. After all, r represents the distance between the center of the two masses, so it will be a unit of distance if we solved the problem correctly.

Keeping Perspective

Roots are yet another “tool” in your mathematical toolbox you can use to help you figure out unknown information from the information you know. We’re able to find unknown information because of the consistent way God holds all things together.

Remember as you solve your problems that although the $\sqrt{\ }$ and fractional exponent notation is defined to mean the positive root for even roots, when we take a square root ourselves, we can’t assume the positive root is the only answer we care about. **When taking the square root of both sides of an equation, be sure to list both the positive and negative answer, unless only one makes sense in the application.**

The blue dots in this field of galaxies, known as the COSMOS field, show galaxies that contain supermassive black holes emitting high-energy X-rays.



4.3 Exploring Inequalities

There are circumstances where we don't need a specific answer; instead, we need a range of answers.

For example, if you're trying to live within a budget, you could spend any amount *less than or equal to* a specific amount.

Or if you need at least a 60 to pass a course, any score *greater than or equal to* a 60 would work.

Or consider a refrigerator. There is energy put into it. Some of that energy goes toward the work of cooling, but some is lost in heat due to inefficiency. The work of cooling is *less than or equal to* the energy put in.

We use inequalities to help us when working with situations with a range of answers. Hopefully, the idea of an inequality is already quite familiar. But we're going to review and look at them in a bit more depth today.

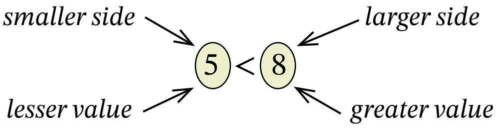
Inequality Symbols

The chart shows some comparison symbols you should know. Keep in mind that these are just agreed-upon symbols to represent the relationship between two expressions — the symbols we use can vary (and have historically).

Symbol	Name
$>$	greater than
$\geq$	greater than or equal to
$=$	equal to

Symbol	Name
$<$	less than
$\leq$	less than or equal to
$\neq$	does not equal

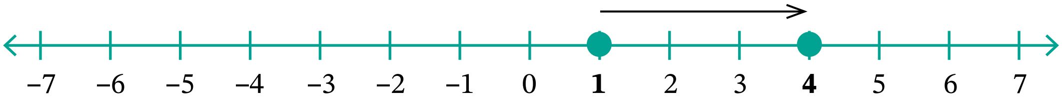
Just remember that the larger side of the inequality goes with the greater value.



A visual way of thinking about inequalities is to use a number line. If a number is farther to the right on a number line than another number, we say it is greater than that number. If it is farther to the left, it is less than.

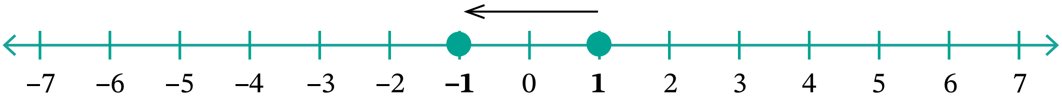
4 is farther to the right than 1, or is greater than 1

$4 > 1$



–1 is farther to the left than 1, or is less than 1

$-1 < 1$



Inequalities and Ranges of Values

We can use inequalities to help us specify the ranges of values an unknown can be. For example, if we know the amount we can spend on groceries for the week must be less than \$100, we could represent the amount our groceries could be like this:

$$g \leq \$100$$

Or if our score must be greater than 60 to pass a test, we could express the ranges of all scores that would give us a passing grade like this:

$$s \geq 60$$

Solving Inequalities

There are times when we have two expressions separated by an inequality. For instance, if the number of hours we work must be greater than or equal to 22 hours to get health insurance and we're working 8 hours every Monday, how many more hours (t) do we need to work? We'd have this:

$$22 \text{ hr} \leq 8 \text{ hr} + t$$

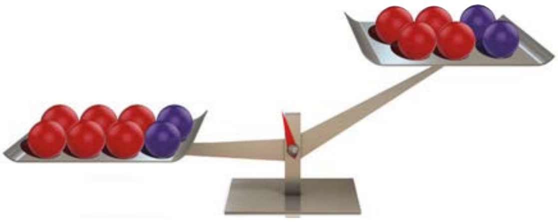
> Just as we did with equalities, we can add, subtract, multiply, or divide the *same amount* to *both sides* of an inequality without changing the meaning of the inequality.

Original Inequality

$$6 \qquad > \qquad 4$$

Added 2 to both sides.

$$6 + 2 \qquad > \qquad 4 + 2$$



Subtracted 2 from both sides.

$$6 - 2 \qquad > \qquad 4 - 2$$

Multiplied both sides by 2.

$6 \cdot 2 > 4 \cdot 2$

Divided both sides by 2.

$6 \div 2 > 4 \div 2$

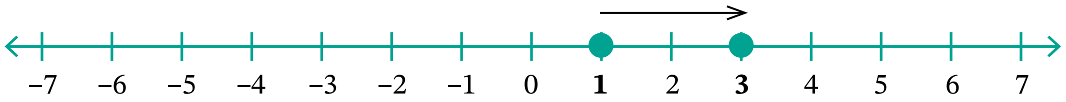
There is one important exception to this with inequalities. **With inequalities, when we multiply or divide by a negative number, we have to flip the direction of the inequality sign for the inequality to hold true.**

After all, when you think about it, we're initially saying that a certain number is further to one side or the other on the number line. When we multiply or divide it by a negative number, however, the negative part takes *the opposite of* the number — so the relationship on the number line ends up flipping. Thus, we have to flip the sign as well.

For example, let's say we have the inequality 3 > 1.

3 is farther to the right than 1.

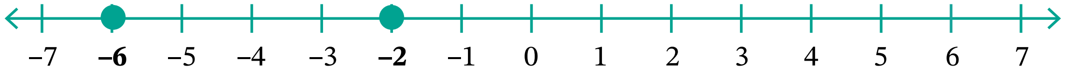
$3 > 1$



If we multiply *both sides* by a negative number, we end up reversing this relationship. For example, let's pick –2. When we do, we get –6 on the left side and –2 on the right.

$$\cancel{(-2)3 > 1(-2)}$$
$$\cancel{-6 > -2}$$

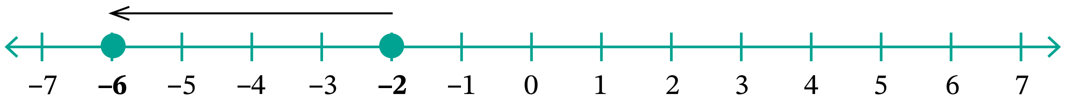
Multiplied both sides by –2.



Notice that –6 is farther to the *left* than –2. So it is *not* true that –6 > –2. Instead, we have to *flip* the sign.

–6 is farther to the left than –2.

$-6 < -2$



In order to preserve the correct relationship between the sides of an inequality, when you multiply or divide both sides by a *negative number* (thus taking *the opposite of* each side), you also have to flip the inequality sign, as which side is greater is now opposite of what it was.

Finding Unknowns in Inequalities

We can use the knowledge that we can add, subtract, multiply, or divide any number to *both sides* of an inequality (being careful to swap the direction of the sign if multiplying or dividing by a negative number) to find unknowns.

Example: Solve $22\text{ hr} \leq 8\text{ hr} + t$ for t.

$14\text{ hr} \leq t$ (subtracted 8 hr from *both sides*)

$t \geq 14\text{ hr}$ (swapped sides — see box)

Notice that when we moved t to the left above we made sure to preserve the meaning by keeping t with the greater side of the inequality. **Be careful when switching what's on each side of the inequality that you also adjust the inequality symbol so that the larger side remains with the greater side.**

$14\text{ hr} \leq t = t \geq 14\text{ hr}$ (both inequalities have the same meaning)

$14\text{ hr} \leq t \neq t \leq 14\text{ hr}$ (the inequalities have different meanings)

Example: Solve $8x > 1$ for x.

$\frac{8x}{8} > \frac{1}{8}$ (divided both sides by 8)

$x > \frac{1}{8}$

Example: Solve $-8x > 1$ for x.

$\frac{-8x}{-8} < \frac{1}{-8}$ (divided both sides by −8 and changed the direction of the sign)

$x < -\frac{1}{8}$

Remember that the direction of the sign changes when multiplying or dividing an inequality by a negative number.

Checking Your Work

As with equalities, you can check your work by plugging the answer you got for the unknown back into the original problem. Only with inequalities, you'll have to think through the > and < signs to see if the problem is correct.

Example: Check if the answer we got of $x < -\frac{1}{8}$ is the correct answer for $-8x > 1$.

Checking inequalities requires more steps, since x could be multiple values. A good place to start is to rewrite the original inequality as an equality and see if the value found would make the equation equal. If it does, that tells us that's the correct value for x to be either greater or less than.

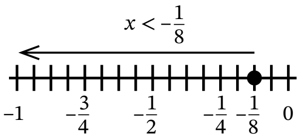
$$\begin{aligned} -8x &= 1 && \text{(rewrote with an equal sign)} \\ -8\left(-\frac{1}{8}\right) &= 1 && \text{(plugged } -\tfrac{1}{8} \text{ in for } x\text{)} \\ 1 &= 1 && \text{(simplified)} \end{aligned}$$

Since we have 1 on both sides, we know that this value is the point at which the inequality would be equal if it were an equality. So we know that $-\frac{1}{8}$ is the point at which numbers would have to be less than or greater than in order to make the expression an inequality. We just have to make sure we found the correct sign.

To check if $<$ was the correct sign, we need to plug in any value *less than* $-\frac{1}{8}$. So let's plug in -1 (the closest whole number less than the $-\frac{1}{8}$ value).

$$\begin{aligned} -8(-1) &> 1 \\ 8 &> 1 \end{aligned}$$

We chose whole numbers to keep the math easy and avoid mistakes. But if $-\frac{1}{8}$ is the correct answer, any number less than $-\frac{1}{8}$ should have resulted in a true inequality, and any number greater than $-\frac{1}{8}$ should have resulted in a false one.



This is a true statement, so we have now verified that $x < -\frac{1}{8}$ does indeed satisfied the inequality. Note we could also plug in a number greater than $-\frac{1}{8}$ to show that it does not work. We could plug in 0 (the closest whole number on the other side) and see

$$\begin{aligned} -8(0) &> 1 \\ 0 &> 1 \end{aligned}$$

which is not true. Thus, we reconfirmed that the inequality solution is indeed the correct one.

Multiplying and Dividing Both Sides of an Inequality by Unknowns

What do we do when we're multiplying or dividing both sides of an inequality by an unknown? After all, we don't know if the unknown is a positive or a negative number! In this case, we have to solve assuming it is positive *and* assuming it is negative and then analyze the problem graphically. It's a skill that's beyond the scope of this course, so we won't be going over it. Just know that **you can't simply multiply or divide both sides of an inequality**

by an unknown without taking into account that it could be either positive or negative.

Keeping Perspective

Inequalities are yet another tool to have in your mathematical toolbox. They give us a way of showing that something has to be greater or less than something else.

One interesting area of study where inequalities come up a lot is in thermodynamics (the study of how heat and energy are transferred). There's always an increase in what we call entropy (represented by the letter S), which is a measure of the order and disorder of the universe. The overall final entropy of the universe is *greater than or equal to* the initial entropy for all exchanges of heat and energy — while entropy can decrease in a local system, it never globally decreases.[4] This basically means that the universe as a whole is growing in its *disorder*.

$$\textit{overall final entropy} \geq \textit{overall initial entropy}$$

or

$$S_{\text{Final}} \geq S_{\text{Initial}}$$

Thermodynamics is a rather in-depth topic with a lot of different aspects to it. How exactly (or if) it operated in the original perfect world is something creationists still debate.[5] It's hard for us as fallen creatures to imagine what a perfect world was like. We believe it's safe to say, though, that the tendency for things to move towards disorder we observe today points to the fact that we live in a fallen world, just as the Bible says.

Random Facts from Dr. Adam

The tendency toward disorder (i.e., the Second Law of Thermodynamics) is one that often comes up when debating creation and evolution. It's too in-depth to cover the arguments here, but the general gist is that this law appears problematic for evolutionists (although they would try to argue otherwise). At the same time, creationists often misrepresent it in their arguments, oversimplifying it. For more details, see "Does the Second Law of Thermodynamics Favor Evolution?" by Dr. Danny Faulkner at https://answersingenesis.org/physics/second-law-of-thermodynamics/.

4.4 Application Problem-Solving Skills

It's important that you know how to apply the various concepts that you're learning. After all, God has given each one of us work to do here on earth. In fact, work was part of the Garden of Eden before the Fall (i.e., in Genesis 2:9, Adam did work in naming all the animals; in Genesis 2:15, he did work in tending the garden)! While the Fall affected everything and made work no longer perfect like God created it, it's still a God-given blessing. And math can aid in that work!

Up until now, application problems have either consisted of plugging numbers into a formula and solving or only required thinking through a few steps. But in real life, problems don't come labeled with what formula to use . . . and they often require more steps to solve.

While being able to apply algebra without some clues as to what formula to use often requires knowledge of physics or financial concepts, you can learn to apply algebra on your own in various settings using basic geometric or everyday concepts you know. And by learning how to think through those problems, you'll be preparing yourself for solving other types of real-life problems too.

Let's look at some pointers to help.

Pointer #1: Don't Panic!

Sometimes, the sight of an unfamiliar problem may be a bit overwhelming. **But don't panic.** Break it down and **see if you can use what you know to help you solve what you don't.**

Pointer #2: Think Before You Start Solving.

Before you begin trying to find the answer, **take the time to think through what you know and what you need to know.** Look for how the information you've been given fits together.

- Start by simply writing out the information you know and the information you need to find.
- Then think through how you can find the information you need to find — it may take more than one step. Here's where you need to look at the information you know and see how it relates to the information you need to find. It's often helpful to use words or symbols to write out equations showing how the information you know relates to the information you're trying to find.

One very easy mistake in solving problems is attempting to jump right into solving the math. But if you haven't properly understood the problem, you could be doing the wrong math! **It's critical that you think through a problem** and make sure that you know what information you've been given, what you need to find, and how to go about finding it before you start performing math operations.

Principles and Methods of Teaching Arithmetic[10] gives 4 helpful steps to problem solving:

1. *Getting a clear understanding of the conditions of the problem*
2. *Planning the solution*
3. *Executing the plan*
4. *Checking the result obtained*

Notice that steps 1 and 2 require thinking *before* you start solving the problem (i.e., executing the plan!) . . . and that checking your work (see Pointer #3) is a step!

Pointer #3: See If Your Answer Makes Sense.

Does the answer make sense? The simple habit of thinking through if the answer you got makes sense can help catch many errors.

Pointer #4: Don't Round Until the End — and Know How to Use Your Calculator.

Sometimes you'll need to perform a lot of calculations before you finish a problem. If you round each time, your ending answer could end up way off. Instead, use the parentheses buttons on your calculator to group numbers and avoid rounding. For example, if solving $\left(1+\frac{1}{6}\right)^2$, either enter 1 ÷ 6 first and then add 1, or enter 1 and then put parentheses around 1 ÷ 6 to tell the calculator to treat that as a separate term being added to the 1. Then square the result. Take a minute to also familiarize yourself with your calculator. Does it have the ability to store an answer in its memory? If so, that can also help you avoid rounding, as it can store the answer for one part of a problem to easily recall back later. As an example, on the TI-83 and TI-84 Plus, there is the "STO →" button. To store a value, type in the value or expression, then the "STO →" button, then the "ALPHA" button, then any button with the letter you want to use to store that value or expression, and then "enter." When you press the "ALPHA" button and that letter again, it will recall whatever you stored there.

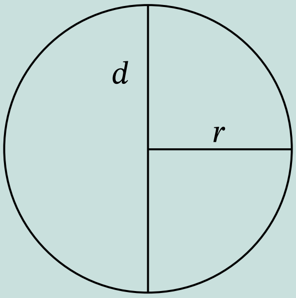
If you don't know how to use the memory function on your calculator, look online for "memory + YOUR CALCULATOR MAKE AND MODEL."

Walking Through an Example

Okay, let's walk through an example problem and see these problem-solving pointers in action.

Circle Math

The example problem below, like many real-life problems you'll encounter, requires remembering various relationships you should have already learned about. In this case, it's going to use information we know about the relationships between the parts of a circle. Note that you can find a list of common geometric formulas in Appendix B: Reference Section.



$$Circumference = \pi d = 2\pi r$$

$$d = 2r$$

Example: If a circus ring is 42 ft in diameter and a horse makes 12 strides each time it goes around it and goes around the ring 3 times, how far does the horse go in each stride? Assume that the path of the horse is 6 inches inside the rim.[6]

What do we know? What do we need to try to find? Below is the information we've been told.

$Diameter_{\text{ring}} = 42 \text{ ft}$

$Strides = 12$

$Path_{\text{horse}} = 6 \text{ in inside rim}$

$Number\ of\ times\ around\ ring = 3$

$distance\ of\ each\ stride = ?$

Note that we could have saved our fingers some writing by using letters rather than words to stand for the distance, strides, etc. The exact letters don't matter — they are just placeholders. Here we used subscripts to help us represent which distance, time, rate, etc., were meant. (The subscript r in d_r reminds us that is for the diameter of the *ring* and d in S_d reminds us it is for the *distance of a stride*.)

$d_r = 42$ ft

$P_h = 6$ in inside rim

$S_d = ?$

$S = 12$

$N = 3$

Now let's plan how to figure out the distance in each stride from this information. If we can figure out the total distance the horse travels in one trip around the ring, we can figure out the distance it travels in a stride by dividing that distance by 12 (the number of strides the horse took to go that distance).

Notice that we're assuming the strides are all the same length — or that we're finding an average stride. To solve real-life problems, we sometimes have to make assumptions like this.

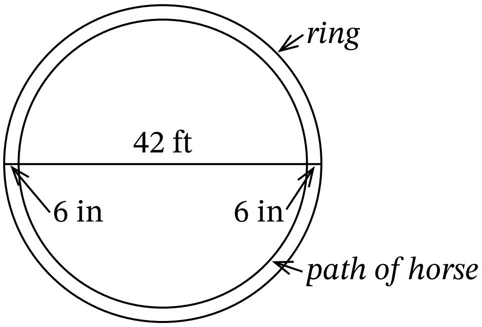
How far did the horse travel in one trip around the ring using those 12 strides? Well, we can use the relationships of a circle that we already know to find that!

$$Circumference\ of\ circle\ horse\ traveled = \pi(diameter\ of\ circle\ horse\ traveled)$$

$$S_d = distance\ of\ each\ stride = \frac{Circumference\ of\ circle\ horse\ traveled}{number\ of\ strides}$$

Now we just need to find the diameter of the circle the horse made. A very easy mistake would be to use 42 ft. But notice that the problem says that the path of the horse is 6 inches *inside* the rim. Another easy mistake to make here would be to use a diameter of 42 ft *minus* 6 in.

However, taking the time to draw the problem out will help us avoid this error.



Take the time to draw out problems involving shapes. It will help you avoid errors.

Notice that if the horse is taking a path 6 inches in from the outside of the ring, the diameter of the circle it is traveling will be 6 inches smaller *on both sides.*

$$\textit{Diameter of circle horse traveled} = 42\text{ ft} - 6\text{ in} - 6\text{ in}$$

Now, another easy mistake to make here would be to simply subtract 42 – 6 – 6. But notice that the 42 is in *feet* while the 6 is in *inches.*

We *can't* add unlike units together. **We have to first make the units the same.**

6 in equals 0.5 ft. We had this memorized, but if we didn't, we could have multiplied by a conversion ratio to see this: $6\text{ in}\left(\frac{1\text{ ft}}{6\text{ in}}\right)$ $= 6\,\cancel{\text{in}}\left(\frac{1\text{ ft}}{12\,\cancel{\text{in}}}\right) = \frac{6}{12}\text{ ft}$ $= \frac{1}{2}\text{ ft} = 0.5\text{ ft}$

$$\textit{Diameter of circle horse traveled} = 42\text{ ft} - 0.5\text{ ft} - 0.5\text{ ft} = 41\text{ ft}$$

The diameter we care about — the one that can help us find the path the horse is traveling — is 41 feet.

Now we can do the math to find the length of each stride.

$$\textit{Circumference of circle horse traveled} = \pi(\textit{diameter of circle horse traveled})$$

$$\textit{Circumference of circle horse traveled} = \pi(41\text{ ft})$$

$$S_d = \textit{distance of each stride} = \frac{\textit{Circumference of circle horse traveled}}{\textit{number of strides}}$$

$$S_d = \textit{distance of each stride} = \frac{\pi(41\text{ ft})}{12} \approx 10.734\text{ ft}$$

Notice that we didn't round until the end in order to preserve as much accuracy as possible.

Now there's one more step: checking to see if our answer makes sense. Does it make sense that a horse can travel 10.734 ft with each stride? Considering how big a horse is, that's probably a reasonable amount. If we had our horse moving forward only 2 ft or as much as 30 ft, though, we might want to recheck our math.

Also, does it make sense that going 10.734 ft for 12 strides would cover $\pi(41\text{ ft})$, or 128.74 ft? Yes, if we round to the nearest whole number, we'd have 11 ft and 12 strides . . . which we can mentally calculate would be approximately 132 ft. Our answer seems reasonable.

Notice that we did not use the information about the number of times the horse goes around the ring that was given. Sometimes in real-life problems, we don't really need all the information we know.

Also notice that we ended up needing to perform several calculations in order to find the requested answer. Real-life problems often take several steps to solve; thinking through what you know and how you can figure out what you don't know upfront is important!

A Reminder About Ratios and Percents

While the problem we walked through assumed a knowledge of circles, many real-life problems assume other knowledge that you should have from previous courses. Here's a quick review of ratios and percents, which will come up on problems in this course.

- As we reviewed in Chapter 1, a ratio is basically a comparison via division. You'll find them coming up in problems dealing with such things as "cost per yard" or "costs a certain amount a yard." Here we're comparing the cost divided by unit. To help solve these problems, set up ratios just like you do for units of measure. Dollars *per* hour can be written $\frac{\text{dollars}}{\text{hour}}$, cost *per* gallon can be written $\frac{\text{cost}}{\text{gallon}}$, etc. For example, if you have 200 lb of a substance that costs \$0.20 cents *per* pound, you can find the total cost like this: $200\ \cancel{\text{lb}}\left(\frac{\$0.20}{\cancel{\text{lb}}}\right) = \$4.00$. Notice how your units canceled out.
- The word *percent* is "short for Latin *per centum,* by the hundred."[7] *Per* actually means "by," and *centum* means "hundred."[8] Thus, 1 percent means "1 per 100." Notice the use of the word *per*. To rewrite 20% as a decimal, just divide it by 100: $\frac{20}{100} = 0.20$. Percents are just a shorthand way of writing comparisons by 100 — that is, ratios with 100 in the denominator!

Keeping Perspective

Work was part of God's original design. While sin has made work harder, work itself is a tremendous, God-given gift. And the problem-solving and study skills you develop in this course can help you in your work, whatever that ends up being.

The Lord God took the man and put him in the garden of Eden to work it and keep it. (Genesis 2:15; ESV)

4.5 Substitution

One incredibly helpful tool we can use to help us solve a problem is called substitution. This tool, as with the ones we've been looking at, helps us complete real-life tasks. Let's take a look.

Understanding Substitution

Let's say you know that the cost of tuition (t) for a semester at a specific university equals \$670 times the number of credits you take (c).

$$t = \$670c$$

Let's say you estimate the cost of textbooks (b) to be \$80 times the number of credits you take (c).

$$b = \$80c$$

Let's also say that your budget for the semester (B) needs to equal the cost of tuition (t) plus the cost of textbooks, plus \$500 for your room and board.

$$B = t + b + \$500$$

Notice from up above that we know what b equals — \$80 times the number of credits. Since b and \$80c represent the *same amount* we can use them interchangeably. We can *substitute* \$80c for b in the $B = t + b + \$500$ equation.

$$B = t + \mathbf{\$80c} + \$500$$

Notice that we're building on the concept of equality. We know if two quantities are equal, then we can use them interchangeably! In other words, we can substitute one for another. This will hold true because of the consistent way God governs all things.

We also know that $t = \$670c$, so guess what? We can *substitute* \$670c for t in the $B = t + \$80c + \$500$ equation.

$$B = \mathbf{\$670c} + \$80c + \$500$$

Now we have an equation to use to compute our budget based on how many credits we decide to take.

We could combine like terms to simplify $B = \$670c + \$80c + \$500$ further to this: $B = \mathbf{\$750c} + \$500$. We'll go over combining like terms in the next chapter.

You may have noticed that the phrase "plugging in" is also used to indicate substituting a value for a value it equals. Both "plugging in" and "substituting" mean the same thing.

Substitute and Insert Values at the End

When solving in-depth real-life problems, it will save you a lot of time (and mistakes!) if you use letters to stand for values up until the final calculation. That way, you don't have to keep track of units of measure every step of the way or perform a bunch of calculations.

Sometimes when exploring God's creation with math we encounter problems that literally take pages to solve. In those situations, waiting to plug values in until the end saves us a *lot* of time and energy . . . especially when those values have complicated units.

Save yourself time down the road and develop the habit of using letters as long as you possibly can in a problem.

Substitution and Units

Have you noticed how the units in many real-life equations get rather cumbersome? Consider the unit for force we get when solving for the force due to gravity produced by 2 objects:

$$\text{kg} \bullet \frac{\text{m}}{\text{s}^2}$$

These units sure aren't fun to keep track of, are they? As we saw in Lesson 2.5, we have a special name for $\text{kg} \bullet \frac{\text{m}}{\text{s}^2}$. We call it a newton (after Isaac Newton, who worked on describing the way God causes objects to attract to each other . . . or what is more commonly called gravity). A newton is abbreviated N.

$$\text{N} = \text{kg} \bullet \frac{\text{m}}{\text{s}^2}$$

Knowing this, we can substitute N for $\text{kg} \bullet \frac{\text{m}}{\text{s}^2}$.

$$90{,}000 \text{ kg} \bullet \frac{\text{m}}{\text{s}^2} = 90{,}000 \text{ N}$$

Notice that we really just used substitution!

Often in physics problems, you'll be given values in newtons. You can then solve the problem using newtons up until the end, at which time you can substitute $\text{kg} \bullet \frac{\text{m}}{\text{s}^2}$ for N to find the correct unit for your answer. For example, look back at the example solved for *r* in Lesson 4.2. We got the equation simplified to this:

$$r = \sqrt{\frac{(1.6675 \times 10^{-9})\left(\frac{\text{m}^3 \bullet \text{kg}}{\text{s}^2}\right)}{\text{N}}} \quad \text{(the equation simplified)}$$

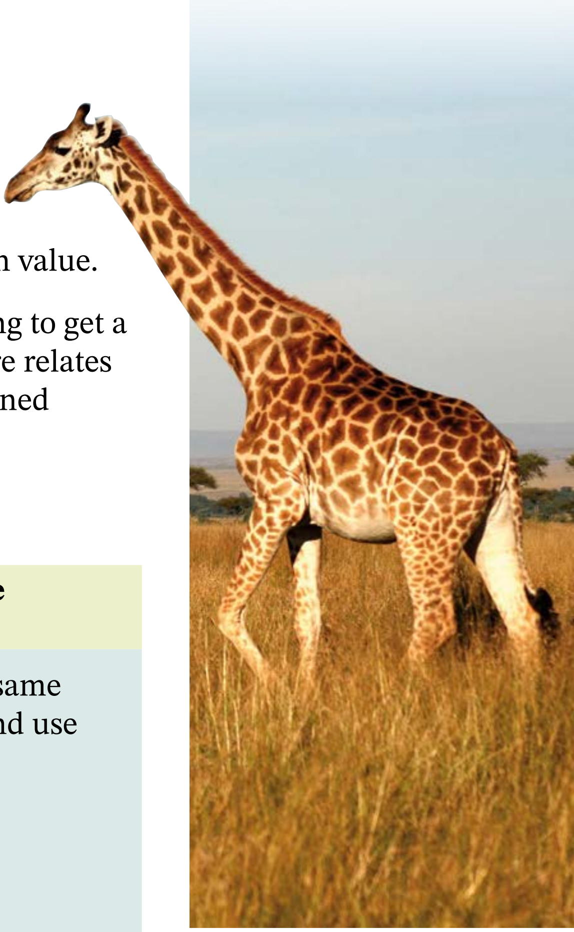
And then substituted $\text{kg} \bullet \frac{\text{m}}{\text{s}^2}$ to finish simplifying the units.

$$r = \sqrt{\frac{(1.6675 \times 10^{-9})\left(\frac{\text{m}^3 \bullet \text{kg}}{\text{s}^2}\right)}{\text{kg} \bullet \frac{\text{m}}{\text{s}^2}}} \quad \left(\text{substituted kg} \bullet \frac{\text{m}}{\text{s}^2}\right)$$

Keeping Perspective

It's easy in upper math to get lost in the mechanics of equation manipulation. Always remember, though, that we manipulate equations *so that* we can solve problems. The main idea behind all the different tools you've learned and will continue to learn in algebra is to use what you know about the consistencies God created and sustains to figure out unknown information. Using multiple equations and substituting a value from one into another is one handy way to do this.

4.6 Chapter Synopsis



In this chapter, we reviewed the basics of solving for unknowns. Be sure to look over the Key Skills and make sure you're comfortable with solving equations and inequalities for an unknown value.

On one of the worksheets that goes with this lesson, you're also going to get a chance to apply what you've been learning to see how blood pressure relates to the distance from the heart . . . and at the amazing way God designed giraffes!

Key Skills for Chapter 4

Be able to solve simple equalities and inequalities for a single unknown.

- Understand that we can perform the same operation using the same value on *both sides* of an equality without changing the value and use this knowledge to solve problems. (Lesson 4.1)

  Examples: $2x + 7 = -2$

  $2x = -9$ (subtracted 7 from both sides)

  $x = \frac{-9}{2}$ (divided both sides by 2)

- Understand the concept of equality, and that $x = 2$ is the same thing as $2 = x$ (we can switch sides of the equals sign without affecting the meaning). (Lesson 4.1)
- Know how to take the same root of *both sides* of an equation to find an unknown, knowing that even roots will have 2 possible answers: a positive and a negative. (Lesson 4.2)

  Examples: $2x^2 = 98$

  $\sqrt{x^2} = \sqrt{49}$ (took the square root of both sides)

  $x = \pm 7$

- Understand that we can add, subtract, multiply, and divide *both sides* of an inequality by the *same positive amount* without changing the value and use this knowledge to solve problems. (Lesson 4.3)

  Examples: $2x - 4 > 8$

  $2x > 12$ (added 4 to both sides)

  $x > 6$ (divided both sides by 2)

- Understand that when multiplying or dividing *both sides* of an inequality by the same *negative amount*, the direction of the sign has to change. (Lesson 4.3)

  Examples: $-2x - 4 > 8$

  $-2x > 12$ (added 4 to both sides)

  $x < -6$ (divided both sides by –2 and changed the direction of the sign)

- Understand that when switching the sides of an inequality, we have to also change the direction of the sign to keep the larger part of the sign with the greater amount. (Lesson 4.3)

  Examples: $8 > x$

  $x < 8$

Know the process to follow to think through word problems. (Lesson 4.4) Remember to not panic, think before you start, see if your answer makes sense, and don't round until the end (and know how to use your calculator).

Understand how to substitute values for unknowns. (Lesson 4.5)

Example: If $x = 5c$ and $4x = 10$, then $4(5c) = 10$.

Chapter 4 Endnotes:

1 Based on *New Oxford American Dictionary,* 3rd ed. (Oxford University Press, 2012), Version 2.2.1 (156) (Apple, 2011), s.v., "algebra."

2 Leonard Euler, *Elements of Algebra, by Leonard Euler, Translated from the French; with the Additions of La Grange, and the Notes of the French Translator* (London: J. Johnson and Co., 1810), https://books.google.com/books?id=hqI-AAAAYAAJ&pg=PR1#v=onepage&q&f=false, p. 270, quoted in Loop, *Principles of Mathematics: Book 2*, p. 176.

3 James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison Wesley, 2003), p. 255-280.

4 See W. Thomas Griffith and Juliet Brosing, *The Physics of Everyday Phenomena: A Conceptual Introduction to Physics*, 6th ed. (New York: McGraw-Hill, 2009), p. 222.

5 See Dr. Danny R. Faulkner, "The Second Law of Thermodynamics and the Curse" from Answers in Genesis (2013), https://answersingenesis.org/physics/the-second-law-of-thermodynamics-and-the-curse/.

6 Based on a problem in Eugene Henry Barker, *Applied Mathematics for Junior High Schools and High Schools* (Boston, MA: Allyn and Bacon, 1920). Available on Google Books, p. 226, http://books.google.com/books?id=-t5EAAAAIAAJ&vq=3427&pg=PR2#v=onepage&q&f=false.

7 *The American Heritage Dictionary of the English Language,* 1980 New College Edition, s.v. "per cent."

8 Ibid.

9 *New Oxford American Dictionary,* 3rd ed. (Oxford University Press, 2012), Version 2.2.1 (156) (Apple, 2011), s.v., "element."

10 James Robert Overman, *Principles and Methods of Teaching Arithmetic* (New York: Lyons and Carnahan, 1920), https://books.google.com/books?id=6gcCAAAAYAAJ&dq=principles%20and%20methods%20of%20teaching%20arithmetic%20overman, p. 256.

Chapter 8

Introduction to Polynomial Functions

8.1 Introducing and Categorizing Polynomials and Polynomial Functions

In the last chapter, we began exploring real-life mathematical relationships that have one output for every value we input. We saw that we call these relationships functions and that there is a lot to explore with functions because God created a complex creation with lots of different types of mathematical relationships.

Now, it's time to begin digging deeper into functions. As we do, we're going to begin using different names to describe functions with specific characteristics. Much like names help us describe specific types of dogs (retrievers, poodles, etc.) and specific types of numbers (negative, positive, imaginary, etc.), we can use names to help us describe specific types of functions. Knowing the type of function a relationship is tells us a lot about the relationship. For example, when I say "poodle," you instantly know some things about the type of dog I'm talking about, right? Knowing the name helped you know a bit about what I was discussing . . . and it will be the same way with different types of functions.

For this chapter and the next, we're going to focus on what we call **polynomial functions.** What is a polynomial function? Before we jump into that, we need to understand what we're meaning by the word polynomial.

Polynomials

We use the word polynomial to describe expressions **that have only positive integer powers of the variables.** In other words, we look at the *variables* in the expression and see if, when the expression is written without any fractions, the variables all have positive exponents. For example, $\frac{x}{5}$ is a polynomial, since x, the variable, has a positive exponent (we can rewrite x

as x^1). However, $\frac{5}{x}$ is *not* a polynomial, since x, the variable, would have a negative exponent if we rewrote it without the fraction:

$$\frac{5}{x} = \frac{5}{x^1} = 5x^{-1}$$

We don't care about the 5 — constants can have either positive or negative exponents. A polynomial *is categorizing expressions based solely on whether the variables have positive integer powers.*

Notice the word "integer" in the definition. This means that d^3 is a polynomial while $V^{\frac{1}{3}}$ is not. This is because 3 is an integer, but $\frac{1}{3}$ is not.

Examples of Polynomials	Example Meaning
$3c$	3 tickets times the cost of each ticket
$\frac{x}{5}$	the total ticket cost divided by the 5 tickets purchased
$\frac{\pi d^3}{6}$	the volume of a sphere with diameter d
$2t + 5$	2 tickets times the cost of each ticket plus a $5 parking fee
$2t + p + 5$	2 tickets times the cost of each ticket plus the cost of a bucket of popcorn plus the $5 parking fee
$2t + 2d + p + 5$	2 tickets times the cost of each ticket plus 2 times the cost of dinner plus the cost of a bucket of popcorn plus the $5 parking fee
$2\pi r^2 + 2\pi rh$	the total surface area of a cylinder with radius r and height h

Examples of Non-polynomials	Example Meaning
$\frac{5}{x}$ This is not a polynomial because when written without a fraction, this would be $5x^{-1}$, and the variable would have a negative exponent.	the total ticket cost of $5 divided by the number of tickets purchased
a^{-8} This is not a polynomial because the variable has a negative exponent.	the probability of rolling a specific value on a die with a sides if you have 8 rolls
$V^{\frac{1}{3}}$ This is not a polynomial because the variable has a non-integer power ($\frac{1}{3}$ is not an integer!).	the length of a side of a cube in terms of its total volume V

The prefix *poly* means "more than one, many, or much,"[1] and some definitions of polynomials say that a polynomial has more than 1 term.[2] Under this definition, then, $\frac{x}{5}$ would not appear to be a polynomial. However, we could always add a 0 to $\frac{x}{5}$ to give the expression a second term: $\frac{x}{5} + 0$. It is common to use the word polynomial to describe expressions **that have only positive integer powers of the variables**, regardless of the number of terms (after all, we could always add terms worth 0 to it). But know that definitions can (and do!) vary.

When determining if an expression is a polynomial, only look at the *variables*. It doesn't matter what the constants are raised to — it only matters if the *variables* have positive integer powers.

Polynomial Functions

Any guesses what a polynomial function is? That's right — **a polynomial function is a fancy way of describing a function that's a polynomial** — that is, a function with only *positive integer powers* in the independent variable (i.e., the input).

The function $y(x) = \frac{7}{x} + x^2$ would *not* be a polynomial function, as there is an x is in the denominator of the first term. If we were to rewrite $\frac{7}{x}$ without the fraction, we'd have x^{-1} . . . a negative power. Similarly, $y(x) = 2x^5 + 3x^{\frac{2}{5}}$ is not a polynomial function since a term has a fractional exponent of $\frac{2}{5}$ and the exponents of variables in polynomials have to all be integers.

We only need to look at the input in functions to determine if it's a polynomial function; it doesn't matter if other unknowns (a, b, etc.) are raised to other powers since in the context of a function other unknowns are treated as constants. For example, $y(x) = 4a^{\frac{1}{2}}x^2$ is a polynomial function even though a is not raised to an integer; it only matters that x (or the input) is raised to a positive integer.

Categorizing Polynomial Functions

Much like we further categorize dogs based on their features, we can further categorize polynomials based on their features. We'll learn 3 different ways to categorize them in this lesson.

Categorizing by the Number of Terms

We have specific names for polynomials with only 1 term (**monomial**), 2 terms (**binomials**), 3 terms (**trinomial**), etc.

You may have also heard polynomials defined as a monomial or the addition or subtraction of 2 or more monomials. This is a different way of defining the same thing. Feel free to remember it however is simplest for you.[3]

These terms apply to polynomials *once they are simplified*. For example, $x + 0$ is a monomial, as it simplifies to x. **Always simplify first before categorizing a polynomial.**

Monomial	**Binomial**	**Trinomial**
polynomial with 1 term when simplified *Examples:* $3c\frac{x}{5}$ $\frac{\pi d^3}{6}$	polynomial with 2 terms when simplified *Examples:* $2t + 5$ $2\pi r^2 + 2\pi rh$ $3x^3 + \frac{x}{\pi}$	polynomial with 3 terms when simplified *Examples:* $2t + p + 5$ $5x^2 + 2x + 3$ $x^5 + \frac{x}{2} + d$

Categorizing by the Highest Degree of the Input

Another way to categorize polynomials is by what we call the highest degree to which the input is raised. **The degree of a function** is simply a fancy way of describing the highest power (i.e., exponent) to which the input is raised. For example, we'd say $f(x) = x^4 + x$ is a fourth-degree polynomial function, since the highest power of x is 4. The chart illustrates this point, using an x to stand for the input.

First-Degree Polynomial Function (i.e., Linear Function)

$f(x) = 5x + 3$

The highest power x is raised to is 1. *Note:* x^1 means the same thing as x, so we don't typically write the 1.

Second-Degree Polynomial Function (i.e., Quadratic Function)

$f(x) = 3x^2 + 2x + 3$

The highest power x is raised to is 2.

Third-Degree Polynomial Function (i.e., Cubic Function)

$f(x) = 5x^3 + 3x^2 + 6x + a^4$

The highest power x is raised to is 3. *Note:* It doesn't matter that a is raised to the 4th power . . . we only care what the highest value the input is raised to.

Fourth-Degree Polynomial Function (i.e., Quartic Function)

$f(x) = 6x^4 + 2x^3 + 5x^2 + 2x + 3$

The highest power x is raised to is 4.

Fifth-Degree Polynomial Function

$f(x) = 6x^5 + 5x^4 + 2x^3 + x^2 + x$

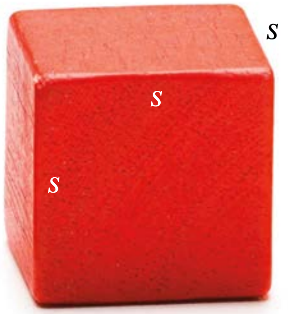
The highest power x is raised to is 5.

. . . and so forth!

You might be wondering why we call a second-degree polynomial function a quadratic function, since we typically associate *quad* with 4, not 2. The answer is because of how quadratic functions help us describe squares. We'll explore this more in the next chapter.

You might notice that some of the polynomial functions have additional names: linear, quadratic, cubic, etc. These are additional names we can use to describe functions with those properties! We'll explore linear functions and quadratic functions more in the next couple of lessons. Hopefully, you've already worked some with them in other math courses.

We can combine naming systems when describing polynomials. For example, we could call $4x + 5$ a first-degree polynomial function . . . and a binomial . . . and a linear function! One naming system describes the highest degree of the input, the other the number of terms, and the other that it forms a line when graphed.



Any guesses why a cubic function is called a cubic function? If you guessed because the equation for finding the volume of a cube based on its side results in a cubic function, you guessed right!

$$V_{\text{cube}} = s^3$$

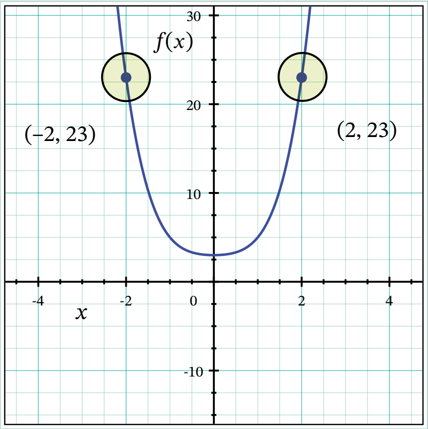
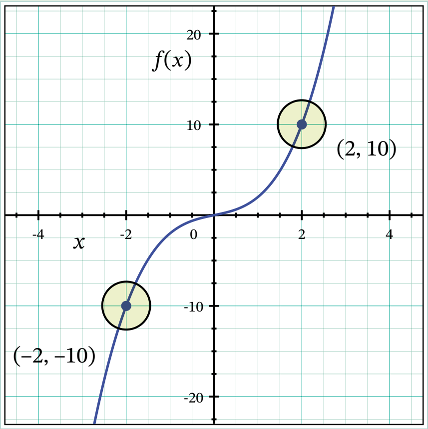
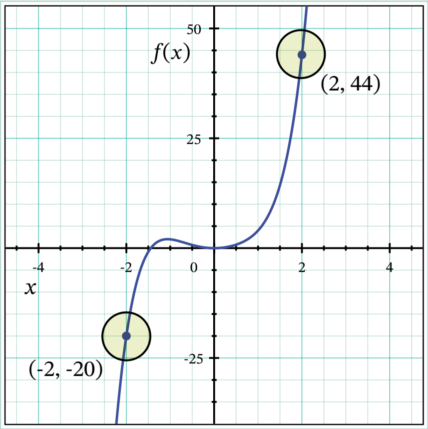
Notice that $V_{\text{cube}} = s^3$ and $f(x) = 5x^3 + 3x^2 + 6x + a^4$ look very different at first glance. But for both of them, the highest power of the input/independent variable (the s in the first function and the x in the second) is 3 . . . so they're both cubic functions!

Categorizing Based on Even or Odd

In the last chapter, we mentioned how functions can be either even, odd, or neither depending on how the outputs for the same negative and positive inputs compared. Well, now that you know about degrees of polynomials, you're ready to learn a shortcut for telling which a polynomial function is. Even and odd is yet a further way of categorizing functions . . . including polynomial functions.

Polynomial functions are even when all the independent variables have even powers, and odd when they have odd powers. If the powers of the independent variable are both even and odd, then the function is neither even nor odd.

Identifying Even and Odd Polynomial Functions

Even Functions	**Odd Functions**	**Neither Odd nor Even**
The output will be the same for the positive input as for the same negative input.	The output for a negative input will be the same absolute value as for the same positive input, *except that the sign will change.*	We can't figure out what the negative values will be based on the positive ones alone, as the output for a positive input and the same negative input are different altogether.
Example: $f(x) = x^4 + x^2 + 3$ $= x^4 + x^2 + 3x^0$ All exponents of the input are even.	*Example:* $f(x) = x^3 + x$ $= x^3 + x^1$ All exponents of the input are odd.	*Example:* $f(v) = v^5 + 3v^2$ $= v^5 + v^2$ All exponents of the input are both even and odd.
		
same output for an input of both –2 and 2	same output, different signs for an input of –2 and 2	totally different outputs for an input of –2 and 2

In determining this, it's important to note the following:

- **Constants count as even powers.** After all, a constant equals the same value, whether x is positive or negative. In fact, a constant could be written as a multiplication by x^0, since x^0 equals 1 and multiplying by 1 doesn't change the value. We could think of 3 as $3x^0$ . . . in which case x has an even power, since 0 can be evenly divided by 2 (i.e., $\frac{0}{2} = 0$).
- **x can be written as x^1, so x has an odd power.**

The rules for figuring out if a polynomial is odd or even makes sense when you think about multiplying negative numbers. When the input has only *odd* powers, then there will be an *odd* number of multiplications . . . which will yield a *negative* result when there's a negative input, and a positive when there's a positive.

On the other hand, an *even* number of multiplications yields a *positive* result for the same negative and positive input. Since the powers tell us how many times to multiply the input by itself, an *even* exponent will always result in a *positive* answer.

Example: Is $f(x) = x^3 + 2x$ odd, even, or neither?

It is odd. Both powers of x (3 and an unwritten 1) are odd.

Example: Is $f(t) = t^6 + 2t^4 + 3t^2 + 5$ odd, even, or neither?

It is even. The exponents 6, 4, and 2 are all even. And even though the last term (the 5) doesn't have a t in it, constants count as even — they give the same output regardless of whether the input is negative or positive. After all, we could rewrite 5 as $5t^0$ without changing the value.

Example: Is $f(x) = 4x^3 + 7$ odd, even, neither?

It is neither. Although the one term with an x has a power of 3 and you would think it is odd, the constant added is considered even.

Keeping Perspective

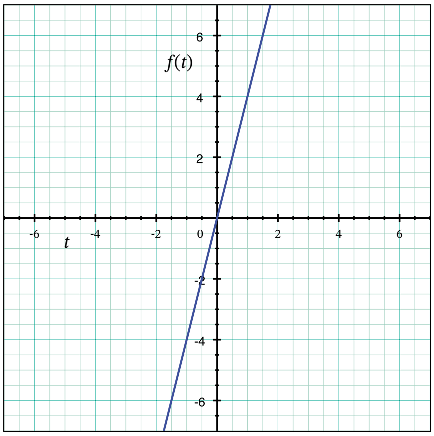
The different ways of describing polynomials we looked at in this lesson are simply useful words to know. We'll use them to help us describe and better explore different types of real-life relationships. The point of learning names to describe specific types of functions is so that we can better understand and describe the real-life mathematical relationships around us . . . consistencies held together by our never-changing, all-powerful God.

Much like Adam named the animals in the Garden of Eden, we're using names to help us categorize and communicate about God's creation. If I tell you that a function is a trinomial fourth-degree function, you instantly know a lot about that relationship — provided, of course, that you know what those words mean.

8.2 Overview of Linear Functions

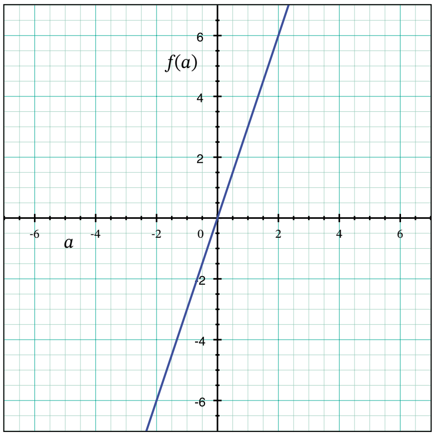
Let's now zoom in further and look at one specific type of polynomial function: linear functions.

Recall from the last lesson that **linear functions** are another name for first-degree polynomial functions — that is, polynomial functions in which the input is raised to the first degree — that is, where the input's highest exponent is 1. The following are all examples of real-life linear functions. Keep in mind that x and x^1 mean the same thing.



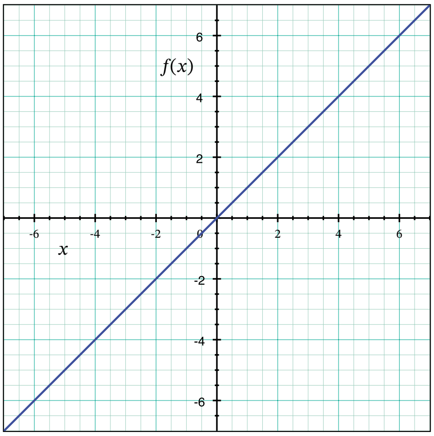
$$d(t) = 4t$$

$$distance = 4(time)$$



$$F(a) = 3a$$

$$Force = 3(acceleration)$$



$$T(x) = x$$

$$Total = x$$

The name linear fits, as when graphed, first-degree polynomial functions **produce a straight line** (notice the *line* in *linear*). They can all be written in this form, which is called the **slope-intercept form**:

$$f(x) = mx + b$$

where m (the slope) and b (the vertical coordinate of the y-intercept) represent constants

You may have seen this written as $y = mx + b$ in previous courses. This means the same thing, except we've used function notation, writing $f(x)$ to represent the output instead of y.

When we refer to the form of a function, keep in mind that the actual letters and values can (and do!) vary. **We're meaning that the equation could be arranged to have this same structure,** where there's an independent variable (shown as x) multiplied by some value (shown as m) with a value (shown as b) added to it.

Notice how we could rewrite each of the examples given above to be in this form.

$$d(t) = 4t + 0 \quad F(a) = 3a + 0 \quad T(x) = 1x + 0$$

All we did was add 0 (which doesn't change the value) and, in the case of $T(x)$, multiply x by 1 (which also doesn't change the value).

Example: Rewrite $g(x) = x$ in slope-intercept form.

We will add a "+ 0" on the right, since adding 0 doesn't change the value.

$$g(x) = x \mathbf{+ 0}$$

And we can write a 1 in front of the x to show multiplying it by 1, since multiplying by 1 doesn't change the value.

$$g(x) = \mathbf{1}x + 0$$

The function is now written in slope-intercept form.

You may sometimes need to simplify in order to rewrite in slope-intercept form.

Example: Rewrite $f(a) = \frac{a^2}{a} + 5a + 3$ in slope-intercept form.

At first glance this might not look like a linear function, but let's simplify.

$$f(a) = \frac{\cancel{a}^2}{\cancel{a}} + 5a + 3$$

$$f(a) = \boldsymbol{a} + 5a + 3$$

$$f(a) = \mathbf{6}\boldsymbol{a} + 3$$

Note that because the original function had a term divided by a, a cannot equal 0.

Remember that you can view subtraction as the addition of a negative number.

Example: Rewrite $f(x) = 2x - 3$ in slope-intercept form.

Even though we have a subtraction with the –3, remember that we can think of subtraction as *addition* of a negative number. We could rewrite this function like this:

$$f(x) = 2x + (-3)$$

Now it is in the form of $f(x) = mx + b$, where $m = 2$ and $b = -3$.

Understanding What Each Value in a Linear Function Does

It can be helpful to understand how the different values in a function affect the function. Knowing that lets us instantly know what would happen to the overall function if a value changed. Let's take a look at how the m and the b — that is, the coefficient of x and the number being added to that product — affect the function.

y-intercept

As we've already explored, the distance you're able to travel equals your speed times your time.

$$d = st$$

So if you are traveling $10\,\frac{\text{mi}}{\text{hr}}$, then the distance is a function of the time:

$$d(t) = \left(10\frac{\text{mi}}{\text{hr}}\right)t$$

But this relationship assumes a starting distance of 0. What if we'd already traveled 5 miles and wanted to see what the total distance would be based on the time we traveled going forward?

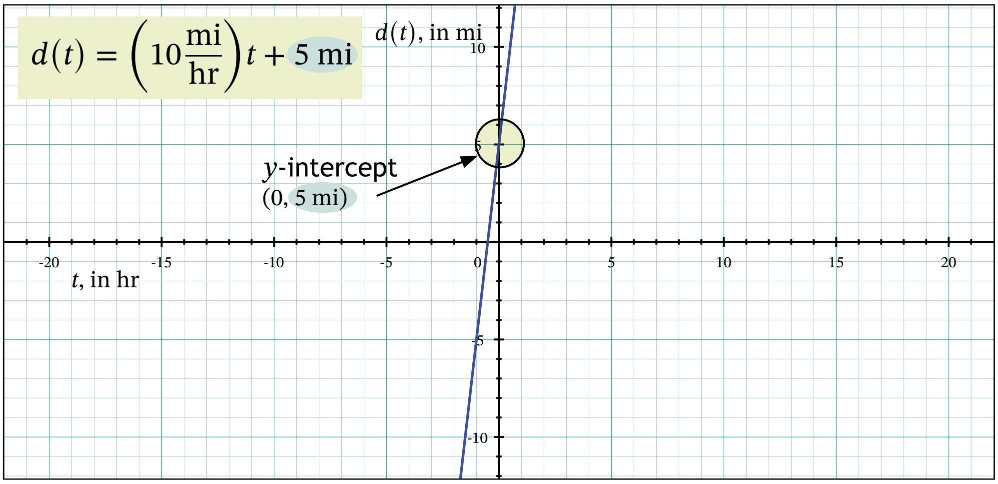
In that case, the total distance would still be the result of multiplying $10\frac{\text{mi}}{\text{hr}}$ and time . . . only we'd then want to *add* the 5 miles we'd already traveled to that.

$$d(t) = \left(10\frac{\text{mi}}{\text{hr}}\right)t + 5\text{ mi}$$

That 5 mi is equivalent to b in the form we looked at earlier.

$$f(x) = mx + b$$

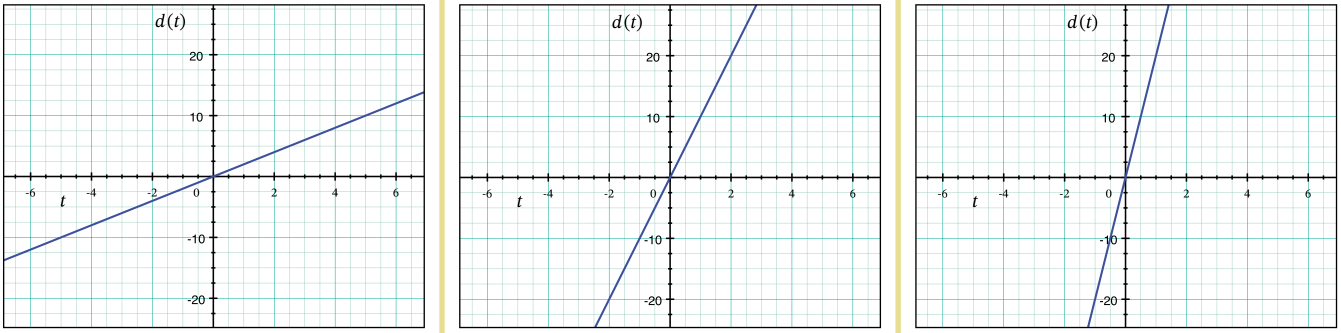
This amount being *added* tells us the starting mileage — that is, the starting value! Not surprisingly, then, it also tells us the vertical coordinate of the y-intercept — that is, what the output will be when the input is 0. After all, when the input is 0, the mx equals 0, as 0 times any number equals 0, leaving us with just the value that we're adding. This is just another way of saying that it's showing the starting value — the output value when the input is 0.



As we saw in the last chapter, we call the point at which a line or curve intersects the y-axis the y-intercept. Since in polynomial functions the value being added corresponds to the vertical coordinate of the y-intercept, **it's common to refer to the constant being added (the b in $f(x) = mx + b$) as just the y-intercept.**

Slope

Now let's talk about the value for the coefficient of the input and how it affects the function. Notice on the following graphs (which show the distance traveled based on time at various speeds) that the line gets more vertical the greater the coefficient of the input (our speed, in this case) gets. That's because for each input, there's a greater output, as it's being multiplied by a greater number. **We call the coefficient of the input (the m in $f(x) = mx + b$) the slope of the function.**



$d(t) = 2(t)$

$d(t) = 10(t)$

$d(t) = 20(t)$

As the coefficient of the input (the m in $f(x) = mx + b$) increases, so does the steepness of the line.

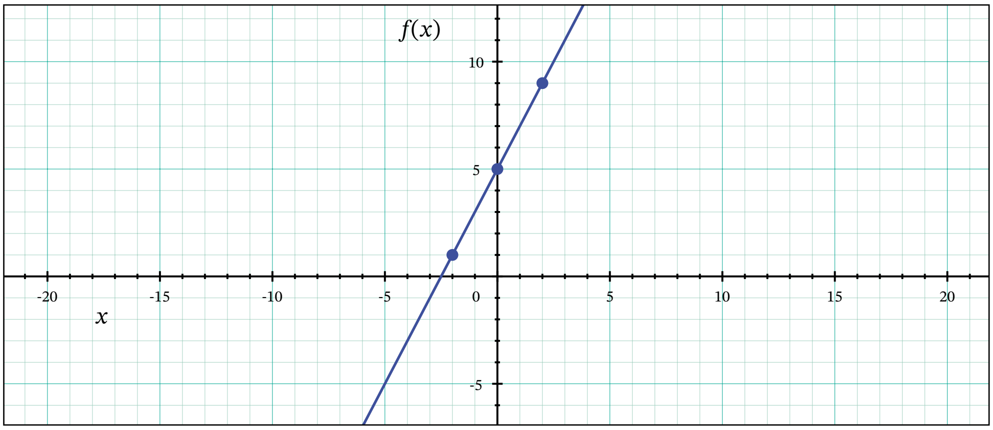
Now you know why $y = mx + b$ is known as **slope-intercept form.** It makes it easy to spot the slope and the y-intercept!

$$y = \underset{\text{slope}}{m}x + \underset{\text{vertical coordinate of the } y\text{-intercept}}{b}$$

Figuring Out the Formula from Data Only

Now, there's a point to trying to understand slopes and intercepts. Sometimes in real life, we don't know the relationship between inputs and outputs. We just have data points that we've collected and know the relationship is linear.

For example, consider the data points shown on the graph, representing a few measurements made of the force and displacement of a spring. We know that the force of a compressed spring and its displacement relate in a linear way. But what function describes this particular line?



Since it relates in a linear way, we know it can be written in the form $f(x) = mx + b$.

Now we know that the y-intercept is 5, because we can see that's where the line intersects the y-axis. Thus we know b must equal 5.

$$f(x) = mx + 5$$

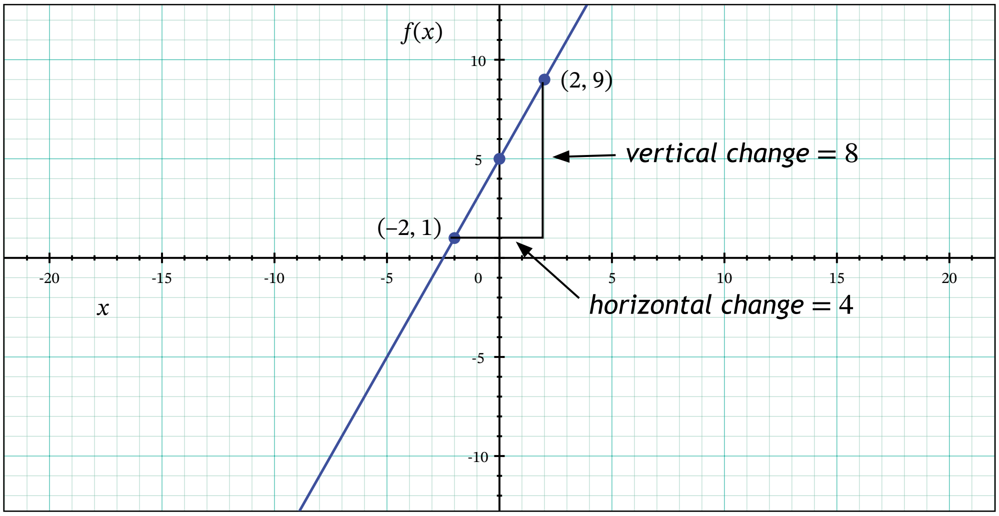
Now we just need to find the m. How do we do that? Well, we can figure it out by looking at the vertical change divided by the horizontal change between any 2 points. We can find this by just visually finding the change between any 2 points on the line or by comparing the coordinates of the points using the formula $\frac{y_2 - y_1}{x_2 - x_1}$.

The word *slope* makes sense to describe the m, as it affects the steepness of the line, just like the slope of a mountain describes its steepness.

Finding the Slope

$$slope = \frac{vertical\ change}{horizontal\ change} \text{ or } \frac{y_2 - y_1}{x_2 - x_1}$$

(The y_2 and x_2 stand for the y and x values of any point on the line, and the y_1 and x_1 for the y and x values of any other point.)



Solving by visually finding the vertical change and the horizontal change and dividing the vertical change by the horizontal:

$$slope = \frac{vertical\ change}{horizontal\ change} = \frac{8}{4} = 2$$

We'll get the same answer if we use any 2 points with the formula $\frac{y_2 - y_1}{x_2 - x_1}$, as the formula finds the vertical change and the horizontal change. **It doesn't matter which point we view as the first and which as the second so long as we are consistent.**

Viewing (2, 9) as the first point (and thus x_1, y_1):

$(x_1, y_1) = (2, 9)$ $(x_2, y_2) = (-2, 1)$ (viewing (2, 9) as the first point and (–2, 1) as the second)

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 1}{2 - (-2)} = \frac{8}{4} = 2$$ (plugged in values into the formula and simplified)

Viewing (–2, 1) as the first point (and thus x_1, y_1):

$(x_1, y_1) = (-2, 1)$ $(x_2, y_2) = (2, 9)$ (viewing (–2, 1) as the first point and (2, 9) as the second)

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 9}{-2 - 2} = \frac{-8}{-4} = 2$$ (plugged in values into the formula and simplified)

It wouldn't have mattered if we had picked 2 completely different points on the line — the ratio between the vertical and horizontal change will be the same throughout since we're dealing with a straight line. Note that this ratio holds true because of the consistent way God created and sustains creation.

We've now found the linear function that describes this line:

$$f(x) = 2x + 5$$

Notice that, provided we know 2 points, we could have found the slope using the formula $\frac{y_2 - y_1}{x_2 - x_1}$ even if we didn't have a graph.

Example: Given the data points (1, 5) and (–2, –4) for an unknown function, find the slope, given that the function is linear.

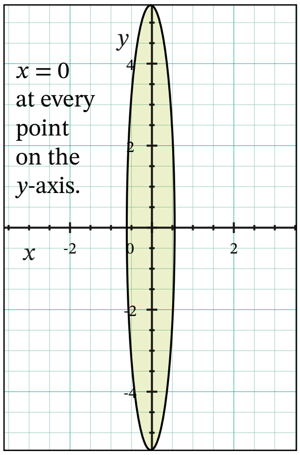
We'll view one point as our first point (x_1, y_1) and the other as our second (x_2, y_2). It doesn't matter which point we view as which.

$$slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{1 - (-2)} = \frac{9}{3} = 3 \text{ or } slope = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - (5)}{-2 - (1)} = \frac{-9}{-3} = 3$$

The slope of a line that goes through both of these points is 3.

But what about the *y*-intercept? Could we have found it without a graph? Yes! Here is the formula for finding the vertical coordinate of the *y*-intercept from data alone:

$$y\text{-}intercept = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$



And we already know the horizontal coordinate of the *y*-intercept — it will always be 0, as it has to be 0 in order for the line to be intersecting the *y*-axis.

Example: Given the data points (1, 5) and (–2, –4), find the *y*-intercept, assuming the function is linear.

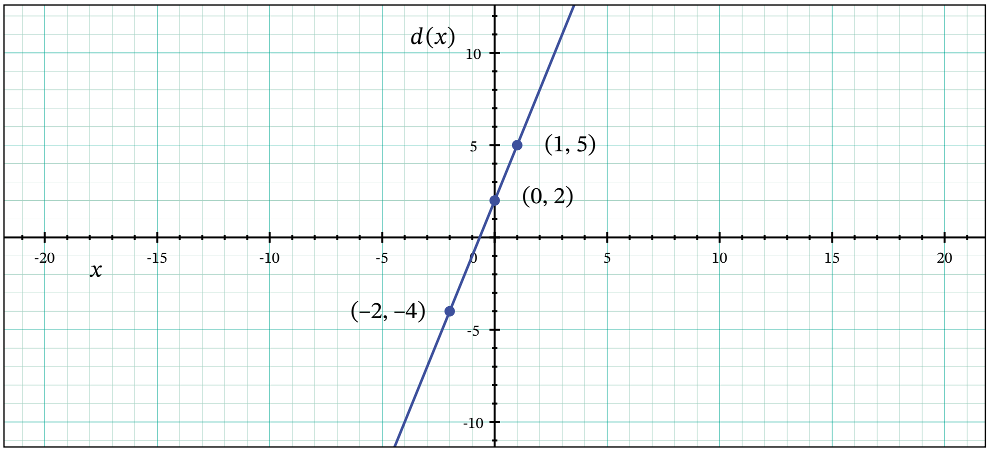
Once again, it doesn't matter which point we view as which, so long as we remain consistent. We'll view (1, 5) as (x_1, y_1) and (–2, –4) as (x_2, y_2).

$$y\text{-intercept} = \frac{y_1x_2 - y_2x_1}{x_2 - x_1} = \frac{5(-2) - (-4)(1)}{-2 - 1} = \frac{-10 - (-4)}{-3} = \frac{-6}{-3} = 2$$

A line that goes through points (1, 5) and (–2, –4) intersects the y-axis when $y = 2$. This means the y-intercept is (0, 2). The formula helped us figure out the output when the input is 0 without having to actually see the graph or know the relationship.

If we put the vertical coordinate of the y-intercept we found in the last example together with the slope we found for those same 2 data points, we have the function: $f(x) = 3x + 2$.

Here's the graph of that function. Notice how the two points we were given are indeed on that graph, and that the y-intercept is at (0, 2).



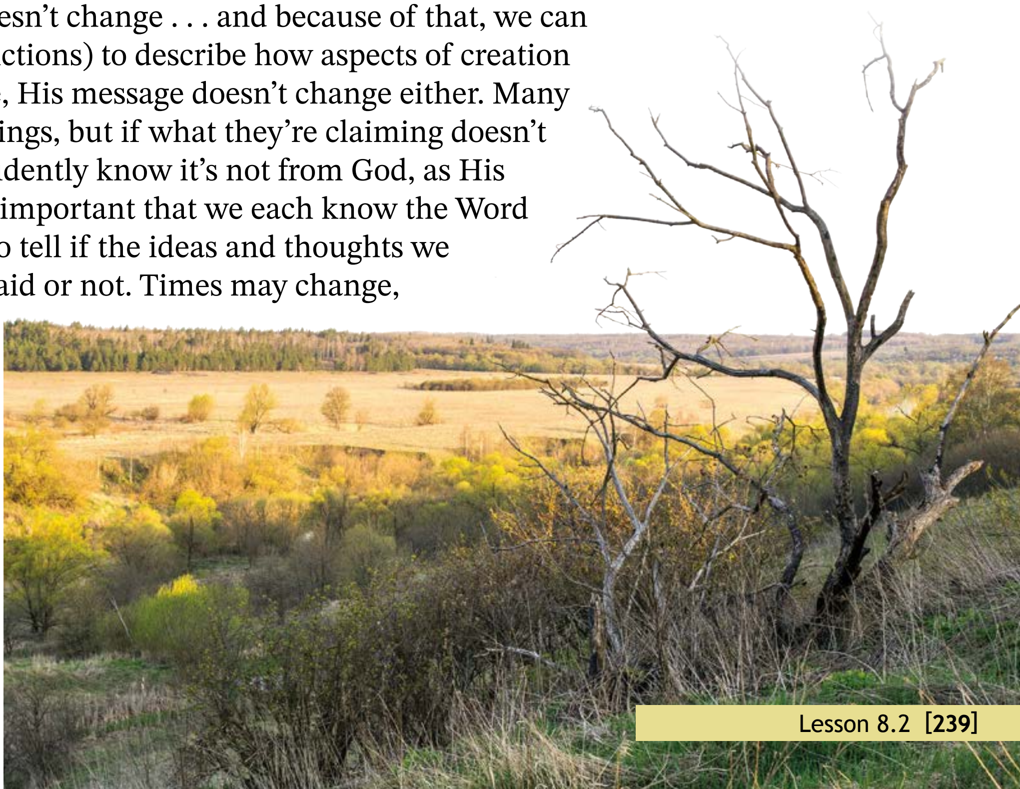
Note that even though we say y-intercept = here, this formula is really finding the *vertical coordinate* of the y-intercept. The horizontal or x coordinate will always be 0, as it has to be 0 in order for the line to be intersecting the y-axis. You should still list the y-intercept as an **ordered pair**. For example, if this formula gives an answer of 4, that means the y-intercept is at the point (0, 4).

Keeping Perspective

Now I know that was a lot of information packed into one lesson. But hopefully you've explored linear functions in the past and are already somewhat familiar with finding y-intercepts and slopes.

Today, ponder the fact that God doesn't change . . . and because of that, we can use functions (including linear functions) to describe how aspects of creation will operate. If God doesn't change, His message doesn't change either. Many people claim God has told them things, but if what they're claiming doesn't match with Scripture, we can confidently know it's not from God, as His message doesn't change. It's super important that we each know the Word and stay in it so that we'll be able to tell if the ideas and thoughts we encounter tie with what God has said or not. Times may change, but His Word and Truth do not. They hold true in every era, just like the functions around us.

The grass withers,
the flower fades,
but the word of our
God will stand forever.
(Isaiah 40:8; ESV)



8.3 Beginning to Look at Quadratics (Minimum/Maximum Point and Vertex Form)

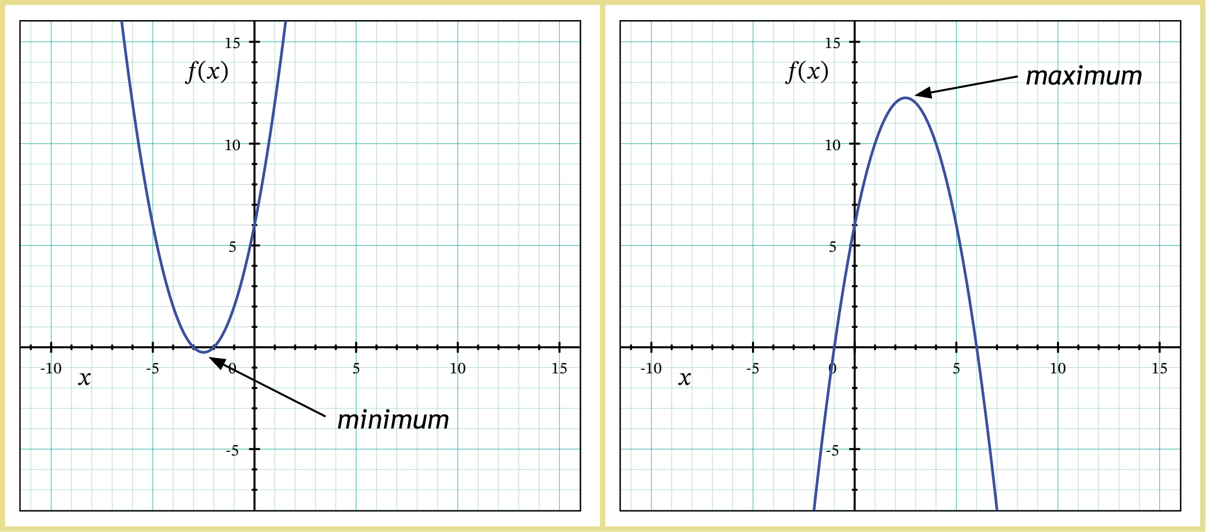
In the last lesson, we reviewed working with linear functions. In the process, we saw that we can describe linear functions using a generalized form $f(x) = mx + b$ and can then use formulas to find the slope (m) and y-intercept (b). Knowing this can help us figure out the equation to describe a line based on just knowing a few points on the line — something that proves quite handy!

As we explore other types of functions, we're also going to learn generalized forms we can use to describe them. And we will encounter various formulas describing how different numbers in the functions relate to values on the graph.

To practice working with functions as well as to prepare for exploring quadratic functions in the next chapter, we're going to look today at how to find the minimum or maximum point of a quadratic function (that is, a polynomial function where the highest exponent of x is a 2).

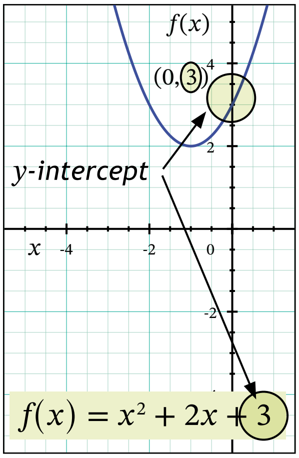
Reviewing Minimum and Maximum

When graphed, quadratics (i.e., 2^{nd}-degree polynomials) form what is called a **parabola** and have either a maximum or a minimum, depending on which direction their curve opens.



It's easy to see on a graph where the minimum/maximum is, but how would we figure it out based on the equation alone? Knowing this will both help us manually graph quadratics (which you'll learn to do in the next chapter). Let's take a look.

Finding the Minimum or Maximum of a Quadratic



Quadratic functions can all be written in this generalized form:

$$f(x) = ax^2 + bx + c, \text{ where } a, b, \text{ and } c \text{ represent constants}$$

Note: c is actually the vertical coordinate of the y-intercept. Think about it. The curve intersects the y-axis whenever $x = 0$. And when $x = 0$, $ax^2 + bx$ would equal 0, as any value multiplied by 0 is 0. So the output would simply be whatever value c equals.

It's important to note that not every quadratic looks like the generalized form to start.

For example, $f(x) = x^2 + 5$ can be rewritten like this:

$$f(x) = 1x^2 + 0x + 5$$

***The feature that makes a function a quadratic is that it's a polynomial where the highest exponent of the input is a 2*. To identify what a, b, and c are, look at the coefficient of x^2, x, and the constant . . . and note that these values may be 1 (or in the case of b and c, 0).**

Example: Identify a, b, and c in $f(x) = x^2 – 2$.

Let's start by rewriting this in the form of $f(x) = ax^2 + bx + c$.

$$f(x) = 1x^2 + 0x + -2$$

Now we can easily see that $a = 1$, $b = 0$, and $c = –2$.

$$f(x) = ax^2 + bx + c$$
$$f(x) = 1x^2 + 0x + -2$$

Now, the input (i.e., horizontal value) at the minimum/maximum point of a quadratic equals $-\frac{b}{2a}$. We'll write that like this:

$$x_{\text{min|max}} = -\frac{b}{2a}$$

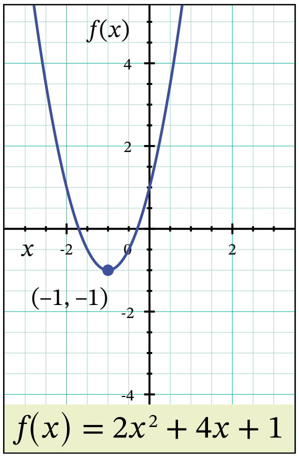
Remember, the a here is referring to the a in $f(x) = ax^2 + bx + c$ — which is representing the coefficient of the x^2. And the b is representing the coefficient of the x.

Knowing this formula and knowing the function is all we need to know to figure out the coordinates of the minimum or maximum point.

Example: Find the minimum or maximum coordinates of $f(x) = 2x^2 + 4x + 1$.

Here, the a, or the coefficient of x^2, is 2, and the b, or the coefficient of x, is 4. We'll plug these values into the formula to find the input at the minimum or maximum point.

$$x_{\text{min|max}} = -\frac{b}{2a} = -\frac{\mathbf{4}}{2(\mathbf{2})} = -\frac{4}{4} = -1$$



The input, or horizontal coordinate, at the minimum or maximum point is −1. But what is the output, or the vertical coordinate?

To find that, we need to plug this input into the function and see what the output at that point will be!

$$f(x) = 2x^2 + 4x + 1 \qquad \text{(original function)}$$

$$f(\mathbf{-1}) = 2(\mathbf{-1})^2 + 4(\mathbf{-1}) + 1 \qquad \text{(plugged in } -1 \text{ for } x\text{)}$$

$$f(-1) = 2(1) - 4 + 1 = -1 \qquad \text{(simplified)}$$

Now we have both our coordinates. The minimum or maximum point will be at $(-1, -1)$.

Notice that if we were to check our work by graphing the function, we'd find this holds true, as shown in the sidebar.

To find the minimum or maximum point of a quadratic written in the form $f(x) = ax^2 + bx + c$, **use the formula** $\boldsymbol{x_{\text{min|max}} = -\frac{b}{2a}}$ to find the input value at the minimum or maximum. That will be the horizontal coordinate. Then plug that value into the function to find the output for that input value. That will be the vertical coordinate.

Note that it doesn't matter that we have t instead of x as the input. The point of the general form $f(x) = ax^2 + bx + c$ is that x represents the input, whatever that is. Likewise, in $x_{\text{min|max}} = -\frac{b}{2a}$, the x represents the input, whatever that is.

Example: Find the minimum/maximum coordinates for $f(t) = t^2 - 2t - 3$.

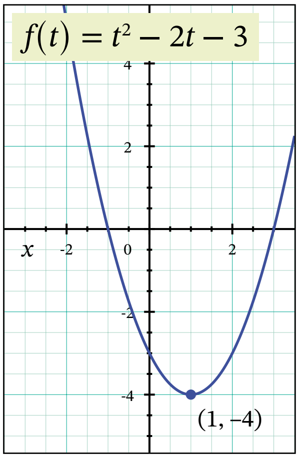
Here, a, the coefficient of t^2, is an unwritten 1 and b, the coefficient of t, is –2. Let's plug these values in to find the input at the minimum or maximum.

Watch your negative signs! Notice that we included the negative sign as part of the value of b. After all, the generalized form of a quadratic ($f(x) = ax^2 + bx + c$) has positive signs in front of bx and c, so **we have to treat any negative signs as part of the value of *b* and *c*.**

Always be careful when working with formulas based on generalized forms of functions that you watch out for your negative signs!

$$t_{\text{min|max}} = -\frac{b}{2a} = -\frac{\mathbf{-2}}{2(\mathbf{1})} = \frac{2}{2} = 1$$

The input at the minimum or maximum point will be 1. Let's plug this value back into the function to find the corresponding output.



$$f(t) = t^2 - 2t - 3 \qquad \text{(original function)}$$

$$f(1) = (\mathbf{1})^2 - 2(\mathbf{1}) - 3 \qquad \text{(plugged in a value of 1 for } t\text{)}$$

$$f(1) = 1 - 2 - 3 = -4 \qquad \text{(simplified)}$$

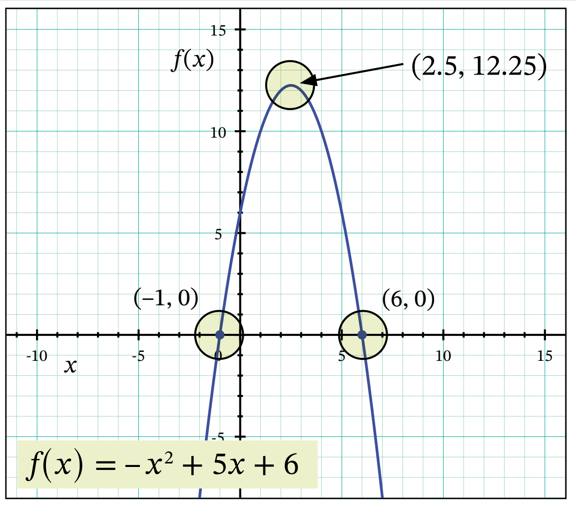
The vertical coordinate at the minimum/maximum is –4, meaning the minimum/maximum occurs at (1, –4).

Notice that if we were to check our work by graphing the function, we'd find this holds true, as shown in the sidebar.

Minimum/Maximum and the x-intercepts/Roots

It's worth pointing out that for quadratics, the minimum/maximum is always located halfway between the quadratic's x-intercepts/roots.

For example, the 2 x-intercepts in the function shown occur when x is 6 and –1. The difference between those 2 points is 7, as $6-(-1)=6+1=7$. Half of that distance is 3.5, as $\frac{1}{2}(7)=\frac{7}{2}$ $=3.5$. If you either subtract 3.5 from 7 or add it to –1 you get 2.5, which is the horizontal coordinate for the maximum point! There's more than one way to find the minimum/maximum point, depending on what information we know. You will learn more about finding the x-intercepts of quadratics in the next chapter.



An Alternate Formula

It's worth noting that quadratic functions can be written in other forms, such as this one (called the **vertex form**; vertex is another term for the minimum/maximum point):

$$f(x)=a\,(x-h)^2+k$$

Note that the a here is the *same* a as in the other form. Only instead of listing b and c, we've listed values called h and k.

When a quadratic is written in this form, it's super easy to find the minimum or maximum: the h is the horizontal value at the minimum/maximum, and the k the vertical one.

Example: What is the minimum/maximum point of $f(x)=2(x-1)^2+3$?

Here, we can see that 1 is in the place where h was in the vertex form, and 3 where the k was. The minimum/maximum occurs at (1, 3).

$$f(x)=a(x-h)^2+k$$

$$f(x)=2(x-1)^2+3$$

Again, though, be sure to watch your positive and negative signs.

Example: What is the minimum/maximum point of $f(x)=5(x+6)^2-2$?

Note that in the vertex form, h is preceded by a negative sign, and k by a positive one.

$$f(x)=a(x-h)^2+k$$

Yet in $f(x)=5(x+6)^2-2$, h is preceded by a positive sign, and k by a negative one. Remember, 2 negative signs simplifies to a positive. We

$$f(x)=a(x-h)^2+k$$

$$f(x)=5(x+6)^2-2$$

$$f(x)=5(x-(-6))^2+-2$$

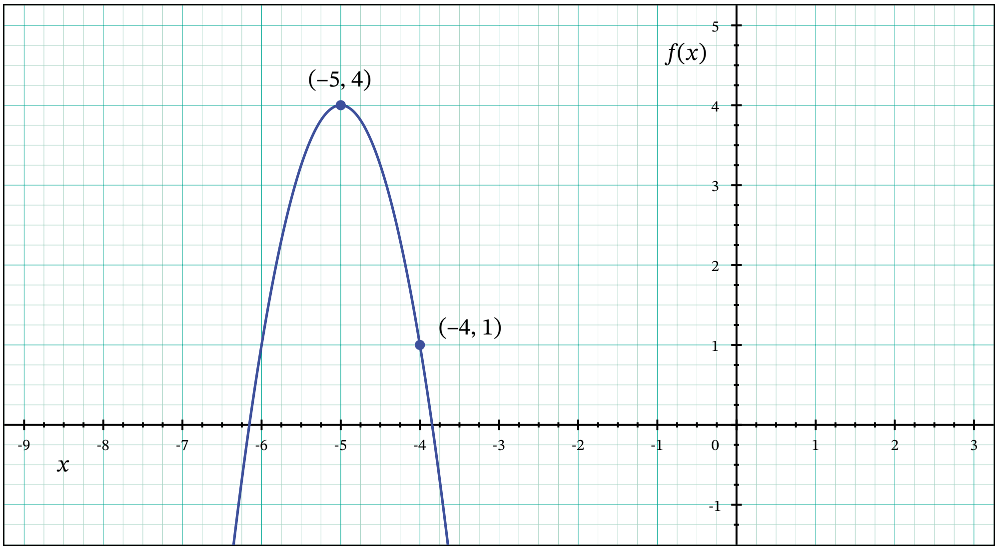
could rewrite + 6 as – (– 6). And we could rewrite –2 as + –2, giving us this:

$$f(x) = 5(x - (-6))^2 + -2$$

Now that it's in the same form as $f(x) = a(x - h)^2 + k$, we can see that h is –6 and k is –2, so the minimum/maximum point is at (–6, –2).

It's worth noting that using the vertex form, you can also write the function to describe a quadratic curve if you just know the minimum/maximum point!

Example: What function describes this parabola?



Parabolas are what quadratic functions look like when graphed. Since we know the maximum point of this parabola, let's plug those values into the vertex form.

$f(x) = a(x - h)^2 + k$	(vertex form)
$f(x) = a(x - (\mathbf{-5}))^2 + \mathbf{4}$	(inserted values)
$f(x) = a(x + 5)^2 + 4$	(simplified)

But wait — what is the value of a? To find that, we need to use another point along the curve. Notice that we know that when the horizontal coordinate (x) equals –4, the vertical coordinate $(f(x))$ equals 1, as the curve passes through (–4, 1). So we can substitute –4 and 1 for x and $f(x)$ in the equation and then solve to find the value for a.

Note that the vertical coordinate here is represented in function notation as $f(x)$. We could rewrite the function as $y = a(x + 5)^2 + 4$. We can insert 1 for $f(x)$ or y because that is the vertical coordinate when $x = -4$.

Thus we have this:

$\mathbf{1} = a(\mathbf{-4} + 5)^2 + 4$ (inserted values for x and $f(x)$ using the other point we were given)

$\mathbf{-3} = a(\mathbf{1})^2$ (simplified inside the parentheses and subtracted 4 from both sides)

$\mathbf{-3} = \boldsymbol{a}$ (simplified)

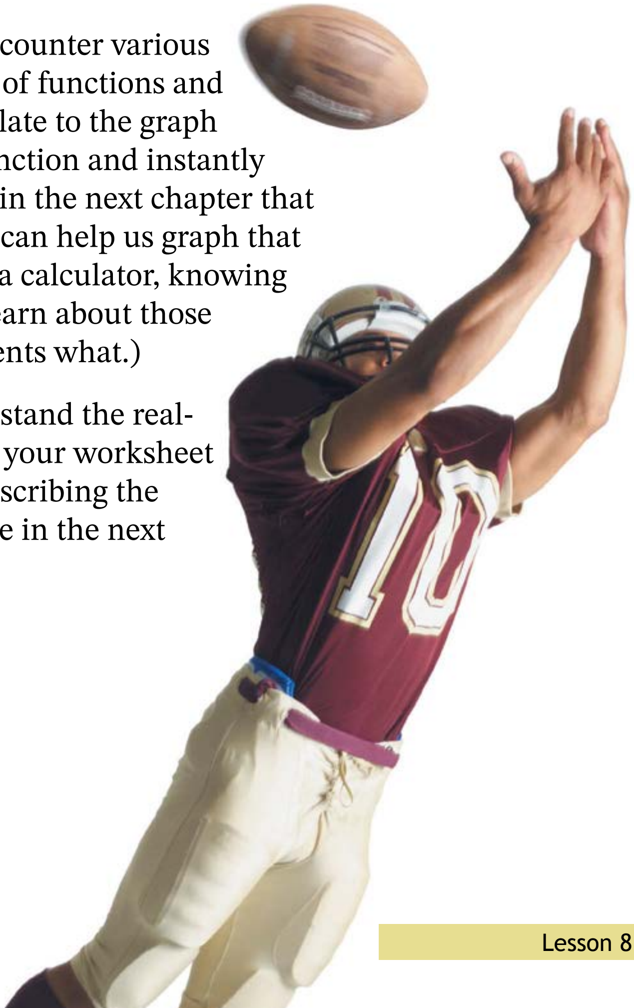
We found the value of a. So the function that describes this curve is this:

$f(x) = \mathbf{-3}(x + 5)^2 + 4$ (plugged in –3 for a)

Note that $f(x) = 5(x + 6)^2 - 2$ can also be written $f(x) = 5x^2 + 60x + 178$ and $f(x) = -3(x + 5)^2 + 4$ can also be written $f(x) = -3x^2 - 30x - 71$. (You can verify this by graphing both forms on a calculator or by multiplying out and simplifying the vertex form — we'll look at how to perform the multiplication in the next lesson.) The general form and vertex form are merely different ways of expressing the *same relationship*.

There are often different ways to express the same relationship. Each one proves useful in different settings. And now you know how to find the maximum and minimum point, regardless of which of these forms you're given a quadratic function in.

Keeping Perspective



As we continue exploring functions, we're going to encounter various formulas that help us generalize about different types of functions and describe how the different numbers in the function relate to the graph of the function. Knowing this can help us look at a function and instantly tell various properties about it. For example, we'll see in the next chapter that finding the minimum/maximum point of a quadratic can help us graph that quadratic by hand. (While graphing is easier to do on a calculator, knowing how to graph common functions by hand helps you learn about those functions and better understand what number represents what.)

The goal is to be better equipped to explore and understand the real-life functions God has placed around us. You'll see on your worksheet several examples of quadratics in action (including describing the path of a ball in the air) — and we'll explore a lot more in the next chapter!

8.4 Working with Polynomial Expressions

For the next several chapters, we're going to continue to look at polynomial expressions. Before we do, though, **let's review some skills we've already looked at,** making sure you know how to apply them to polynomial expressions.

Distributing Multiplication with Multiple Expressions in Parentheses

Back in Lesson 5.5, we looked at what is referred to as the distributive property: a way of describing something true about multiplication that is true because of how God governs all things.

The Distributive Property of Multiplication

If you distribute multiplication among terms in an expression and then add those results together, you'll get the same result as if you first did the addition and then multiplied by the sum.

$$a(b+c) = ab + ac$$

So far, all of your problems related to the distributive property have only involved 2 expressions. You've learned in problems such as the one below to distribute each term of the first expression in parentheses by each term of the second.

$$(x+2)(x-2) = x(x-2) + 2(x-2) = x^2 - 2x + 2x - 4 = x^2 - 4$$

As we continue exploring polynomial functions, we're going to find ourselves with more than 2 expressions, though, like this:

$$4(x+2)(x-2)$$

Or even like this:

$$(x^2-2x-3)(x+2)(x-2)$$

How do we apply the distributive property in these cases? Let's take a look.

Example: Distribute the multiplication in $4(x+2)(x-2)$.

> We can solve this by distributing one term at a time. We'll distribute the 4 across the first parentheses.
>
> $$4(x+2)(x-2) = (4x+8)(x-2)$$

Notice that we did not multiply the 4 by both expressions in parentheses — just one of them. To understand why, think about it with numbers for a minute. Say x equals 0 and the first expression in parentheses ends up equaling 2 and the second -2. We'd have this:

$$4(2)(-2)$$

If we multiplied the 4 by *both* the 2 and the –2 and then multiplied that product by –2, we'd get the wrong answer! We know we need to multiply 4 by 2, getting 8, and then multiply 8 by –2, thereby only multiplying by each factor once.

$$8(-2)=-16$$

When we distributed the 4 across the first parentheses, we were likewise multiplying the 4 by *one* of the other factors (the first expression in parentheses).

$$4(x+2)(x-2)=(4x+8)(x-2)$$

Just to show that this did indeed work, notice that if we plug the same value in for x (we'll use 0) to both sides of the above equation, we get the same answer.

$4(x+2)(x-2)=(\mathbf{4x+8})(\mathbf{x-2})$	(left side is the original expression; right-hand side is the result of distributing the 4 across one expression in parentheses)
$4(\mathbf{0}+2)(\mathbf{0}-2)=(4(\mathbf{0})+8)((\mathbf{0})-2)$	(substituted 0 for x)
$4(\mathbf{2})(\mathbf{-2})=(\mathbf{8})(\mathbf{-2})$	(simplified inside the expression in parentheses)
$\mathbf{-16=-16}$	(simplified)

We can then continue distributing the multiplication across the next term:

$$(4x+8)(x-2)=4x(x-2)+8(x-2)=4x^2-8x+8x-16$$

And now we can simplify by combining like terms:

$$4x^2-16$$

Because multiplication is commutative and associative, it doesn't matter in what order we distribute the multiplication. We could have multiplied the 4 by $(x-2)$ instead of by $(x+2)$. The point is to just distribute it across 1 term, as that way we've accounted for that multiplication.

Note that you can check to see if your answer is correct by substituting a real number for the unknown in both your distributed expression and the original. If you did they math correctly, they should both simplify to the same value!

Original expression when $x=1$:

$4(x+2)(x-2)$	(original expression)
$4(\mathbf{1}+2)(\mathbf{1}-2)=4(3)(-1)=-12$	(evaluated when $x=1$)

Distributed expression when $x = 1$:

$$4x^2 - 16 \qquad \text{(distributed expression)}$$

$$4(\mathbf{1})^2 - 16 = 4 - 16 = -12 \qquad \text{(evaluated when } x = 1\text{)}$$

The expressions equal when evaluated for the same value of x.

You can **substitute a real number for an unknown** to check yourself and see if 2 expressions really are equal.

Example: Distribute the multiplication in $(x^2 - 2x - 3)(x + 2)(x - 2)$.

Since order doesn't matter, let's multiply the $(x + 2)(x - 2)$ first, as they have fewer total terms:

$$(x^2 - 2x - 3)(x + 2)(x - 2) = (x^2 - 2x - 3)(x^2 - 2x + 2x - 4)$$

Here's how we multiplied $(x + 2)(x - 2)$:

$$(x + 2)(x - 2)$$

$$= x(x - 2) + 2(x - 2)$$

$$= x^2 - 2x + 2x - 4$$

Only as you get more comfortable, you can do this step in your head. All you're doing is multiplying the 1st term in the first parentheses by each term in the 2nd parentheses, and then multiplying the 2nd term in the first parentheses by each term in the 2nd parentheses.

Now let's simplify by combining like terms:

$$(x^2 - 2x - 3)(x^2 - 4)$$

Now we just multiply these two expressions in parentheses together, making sure we distribute each term:

$$(x^2 - 2x - 3)(x^2 - 4) = x^2(x^2 - 4) + -2x(x^2 - 4) + -3(x^2 - 4)$$
$$= x^4 - 4x^2 - 2x^3 + 8x - 3x^2 + 12$$

And now we can simplify:

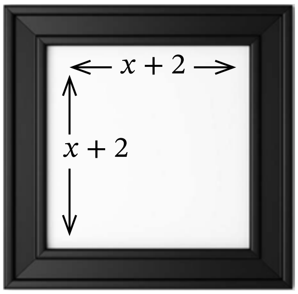
$$x^4 - 2x^3 - 7x^2 + 8x + 12$$

While we won't show it here, we could substitute a value for x to check our work.

When doing complicated distribution like this, just make sure you multiply *each term* of one expression in parentheses by *each term* of the other. Draw arrows between them if it helps you remember the ones you have multiplied as you go.

Squaring Polynomials

Let's say that a square has sides that are some value plus 2, or $x + 2$. The area then would equal that value squared, or $A = (x + 2)^2$.



If we wanted to actually square $x + 2$, we'd need to remember that squaring means multiplying by itself. We *don't* want to just multiply x by itself and 2 by itself. Instead, we want to **multiply the entire term by itself.**

$$(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$$

Example: Distribute the multiplication in $\left(3t + \frac{3}{4}\right)^2$.

Again, let's just remember that squaring means multiplying *the entire term* by itself.

$$\left(3t + \frac{3}{4}\right)\left(3t + \frac{3}{4}\right) = 9t^2 + \frac{9}{4}t + \frac{9}{4}t + \frac{9}{16} = 9t^2 + \frac{18}{4}t + \frac{9}{16} = 9t^2 + \frac{9}{2}t + \frac{9}{16}$$

Note that we multiplied each term in the first parentheses by each term in the second. We could have rewritten this out to clarify: $3t\left(3t + \frac{3}{4}\right) + \frac{3}{4}\left(3t + \frac{3}{4}\right)$.

In the last lesson, we looked at the vertex form and mentioned that it could be rewritten in the general form. You now know all you need to in order to do so!

Example: Rewrite $f(x) = -3(x + 5)^2 + 4$ in the general form.

To rewrite this function (which is one of the vertex form quadratic examples we looked at in the last lesson), we just need to complete the multiplication on the right side! We'll ignore the $f(x)$ for now and just write the right-side of the function.

To start with, we need to remember that $(x + 5)^2$ means $(x + 5)(x + 5)$.

$$-3(x + 5)(x + 5) + 4$$

Now, we can distribute in whatever order we choose. We'll go ahead and distribute the $(x + 5)$ by the $(x + 5)$.

$$-3(x + 5)(x + 5) + 4 = -3(x^2 + 5x + 5x + 25) + 4$$

Now let's combine like terms inside the parentheses.

$$-3(x^2 + 10x + 25) + 4$$

Now let's distribute the –3.

$$-3(x^2 + 10x + 25) + 4 = -3x^2 - 30x - 75 + 4 = -3x^2 - 30x - 71$$

We've now rewritten the right side of the function in the general form. We just need to add back $f(x)$ on the left.

$$f(x) = -3x^2 - 30x - 71$$

Keeping Perspective

As problems get more terms to them, remember that the *same principles you already know apply*. The consistencies God created and sustains operate the same way in complicated problems as they do in simple — just think through what is happening and break them down step by step. And remember that you can **substitute a real number for an unknown** to check yourself and see if 2 expressions really are equal.

The worksheets that go with this lesson will help you review some key concepts you'll need to remember as we continue exploring polynomials. Remember, these skills help us describe the complexities of God's creation and complete the tasks He's given us to do, as we'll see more of as we continue.

8.5 Chapter Synopsis

Congratulations! You've now gotten a thorough introduction to functions and a closer look at polynomial functions in particular. For much of the second half of this course, we're going to be continuing to explore functions in more depth, learning more about the incredible mathematical relationships God created and sustains all around us.

Key Skills

Know what a polynomial is and terms we use to describe polynomials based on their attributes. (Lesson 8.1)

- A polynomial is an algebraic expression with only *positive integer powers in the variables.* For example, $\frac{x}{2}$ is a polynomial, while $\frac{2}{x}$ is not, as it could be rewritten $2x^{-1}$. The tables show more specific labels for polynomials with specific properties.

Labeling Polynomials Based on the Number of Terms

One Term: Monomial	$5x$
Two Terms: Binomial	$5x + 2$
Three Terms: Trinomial	$7x^2 + 5x + 2$

Labeling Polynomials Based on the Highest Power

Highest Power is 1: Linear or First-Degree	$5x^1$ or $5x$
Highest Power is 2: Quadratic or Second-Degree	$5x^2$
Highest Power is 3: Cubic or Third-Degree	$5x^3$
Highest Power is 4: Quartic or Fourth-Degree	$5x^4$
Highest Power is 5: Fifth-Degree	$5x^5$

. . . and so forth!

Note that a polynomial could be several things at once — the function $y = 5x$ is a monomial, linear, and first-degree polynomial function!

Understand how to tell if a polynomial function is even or odd. If the input has only odd powers, then the function will be odd (and x counts as odd — it can be written x^1). Similarly, if all the input has only even powers (and 0 counts as an even power, so constants, which can be written as multiplications by x^0 without changing their value, count as even powers), then the function will be even. If the input has both even and odd powers, the overall function will be neither. (Lesson 8.1)

Use various formulas to describe linear functions from either the data or a graph. (Lesson 8.2)

Know how to work with generalized forms of functions and formulas, such as how to find the minimum or maximum point of a quadratic using the formula $x_{\text{min|max}} = -\frac{b}{2a}$, knowing that quadratics can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c represent constants, and in the form $f(x) = a(x - h)^2 + k$ (called the vertex form), where h is the horizonal value at the minimum/maximum point and k is the vertical value at that point. Also know how to write quadratic functions in vertex form if given the minimum/maximum point and one other point on a **parabola** (the curve a quadratic function forms). (Lesson 8.3)

Know how to apply the distributive property when multiplying 3 or more expressions. (Lesson 8.4)

Example: Distribute the multiplication in $4(x + 2)(x - 2)$.

$(4x + 8)(x - 2)$	(distributed the 4 by each term in the first parentheses)
$4x^2 - 8x + 8x - 16$	(distributed each term in the first parentheses by each term in the second)
$4x^2 - 16$	(combined like terms)

Know how to simplify squared polynomials. (Lesson 8.4)

Example: $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

Chapter 8 Endnotes:

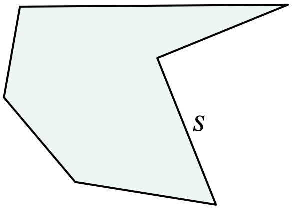
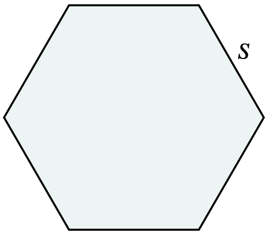
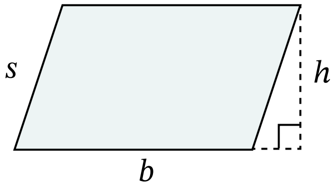
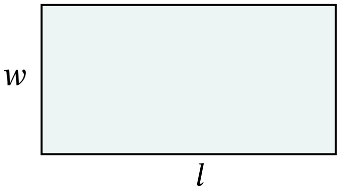
1 *The American Heritage Dictionary of the English Language*, 1980 New College Ed., s.v., "poly–."

2 "An algebraic function of two or more summed terms . . . " Ibid., s.v., "polynomial." See also the definition given in Henry Lewis Rietz, Arthur Robert Orathorne, *School Algebra: Book 1*, ed. Edson Homer Taylor (New York: Henry Holt, 1915), p. 53.

3 We defined a polynomial, and then defined a monomial based on that definition. In *Elementary Algebra*, Jacobs defines a monomial first, and then defines a polynomial as "either a monomial or an expression indicating the addition and/or subtraction of two or more monomials." Harold R. Jacobs, *Elementary Algebra* (Master Books, Green Forest, AR: 2016), p. 336.

Appendix B
Reference Section

Geometric Formulas

Shape Name	Type of Shape	Perimeter	Area
Polygons *closed, two-dimensional figure with straight lines*		$P =$ *sum of all side lengths* $P = s_1 + s_2 + s_3 + \ldots + s_n$	View as multiple triangles or other simple polygons.
Regular Polygon *All sides equal and all angles congruent.*		$P = (\textit{number of sides}) \times (\textit{length of a side})$ $P = ns$	View as multiple triangles or other simple polygons.
Parallelogram *four-sided polygon with both pairs of opposite sides parallel*		$P = 2b + 2s$	$A = bh$
Rectangle *parallelogram with right angles*		$P = 2l + 2w$	$A = lw$

$P =$ *perimeter*

$B =$ *area of the base*

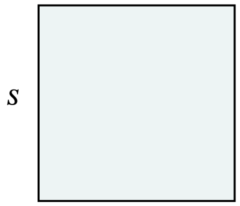
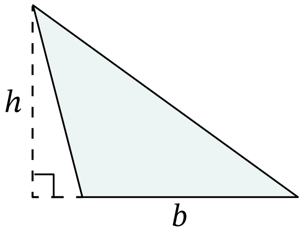
$r =$ *radius*

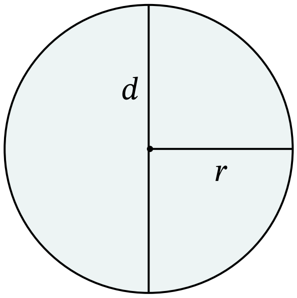
$C =$ *circumference*

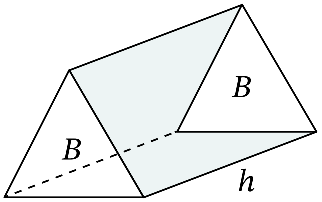
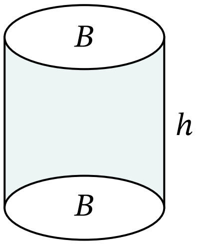
$C_{\text{base}} =$ *circumference of base*

$d =$ *diameter*

$A =$ *area*

Shape Name	Type of Shape	Perimeter	Area
Square *parallelogram with equal-length sides and right angles*		$P = 4s$	$A = s^2$
Triangle *three-sided polygon*		$P =$ *sum of all side lengths* $P = s_1 + s_2 + s_3$	$A = \frac{bh}{2}$

Shape Name	Type of Shape	Circumference, Diameter, and Radius	Area
Circle *closed, two-dimensional figure; each part of the edge is equally distant from the center*		$C = \pi d = 2\pi r$ $d = 2r$ $r = \frac{1}{2}d$	$A = \frac{\pi d^2}{4}$ $A = \pi r^2$

Solid Name[1]	Type of Solid	Volume	Area
Prism *solid with two bases that are parallel polygons and faces (sides) that are parallelograms*		$V = Bh$	$A_{\text{surface}} = 2B +$ *Area of each side*
Cylinder *Solid with two bases that are equal parallel circles, having an equal diameter in any parallel plane between them*		$V = Bh$	$A_{\text{surface}} = C_{\text{base}}h + 2B$

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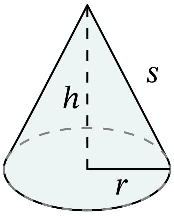
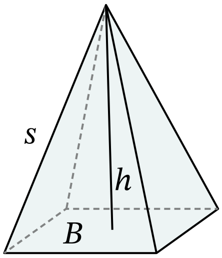
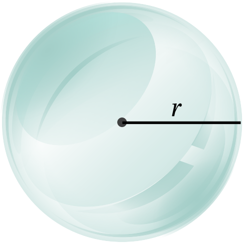
[1] Definitions of solids were based on *Ray's New Higher Arithmetic*, Revised (Cincinnati: Van Antwerp, Bragg & Co., 1880), p. 390-391.

Solid Name	Type of Solid	Volume	Area
Cone *solid whose base is a circle, and whose other surface comes to a common vertex*		$V = \pi r^2 \frac{h}{3}$	$A_{\text{surface}} = \pi rs + \pi r^2$
Square Pyramid *solid with square base whose faces are triangles with a common vertex*		$V = \frac{1}{3}Bh$	$A_{\text{surface}} = B +$ *sum of area of all faces* (sides) Or $A_{\text{surface}} = B + 2s\sqrt{B}$
Sphere *solid bounded by a curved surface, every point of which is at the same distance from the center*		$V = \frac{4}{3}\pi r^3$	$A_{\text{surface}} = 4\pi r^2$

$P =$ *perimeter*

$B =$ *area of the base*

$r =$ *radius*

$C =$ *circumference*

$C_{\text{base}} =$ *circumference of base*

$d =$ *diameter*

$A =$ *area*

Units of Measure

Area – Other

1 acre = 43,560 ft^2

Capacity – Dry

- *U.S. Customary*

2 pints (pt) = 1 quart (qt)

8 quarts = 1 peck (pk)

4 pecks = 1 bushel (bu) = 32 quarts (qt)

- *Conversion Between Systems*

1 quart ≈ 67.201 inches^3

1 bushel = 2,150.420 inches^3

Note: The pint and quart here represent a larger capacity than the ones measuring liquid — they should not be used interchangeably. Unless the problem specifically states otherwise, you can assume pint and quart in this course refer to the liquid units.

Capacity – Liquid

- *U.S. Customary*

3 teaspoons (tsp) = 1 tablespoon (Tbsp)

16 tablespoons = 1 cup (c)

2 cups = 1 pint (pt)

2 pints = 1 quart (qt)

4 quarts = 1 gallon (gal)

2 tablespoons (Tbsp) ≈ 1 fluid ounce (fl oz)

8 fl oz = 1 cup (c)

16 fl oz = 1 pint (pt)

32 fl oz = 1 quart (qt)

128 fl oz = 1 gallon (gal)

- *Metric*

10 milliliters (ml *or* mL) = 1 centiliter (cl *or* cL)

10 centiliters = 100 milliliters = 1 deciliter (dl *or* dL)

10 deciliters = 100 centiliters = 1,000 milliliters = 1 liter (l *or* L)

10 liters = 1 dekaliter (dal *or* daL)

10 dekaliters = 1 hectoliter (hl *or* hL)

10 hectoliters = 1,000 liters = 1 kiloliter (kl *or* kL)

- *Conversion Between Systems*

1 teaspoon ≈ 4.929 milliliters

1 gallon ≈ 3.785 liters

1 pint = 28.875 in^3

1 quart = 57.75 in^3

1 gallon = 231 in^3

Distance

- *Distance – U.S. Customary*

  12 inches (in) = 1 foot (ft)
  3 feet = 36 inches = 1 yard (yd)
  1,760 yards = 5,280 ft = 1 mile (mi)

- *Distance – Metric/SI*

  10 millimeters (mm) = 1 centimeter (cm)
  10 centimeters = 1 decimeter (dm)
  10 decimeters = 100 centimeters = 1,000 millimeters = 1 meter (m)
  10 meters = 1 decameter (dam)
  10 decameters = 1 hectometer (hm)
  10 hectometers = 1,000 meters = 1 kilometer (km)

- *Conversion Between Systems*

  1 inch (in) = 2.540 centimeters (cm)
  1 foot (ft) = 30.480 centimeters (cm)
  1 yard (yd) ≈ 0.914 meter (m)
  1 mile (mi) ≈ 1.609 kilometers (km)

Mass

- *U.S. Customary*

  $$1\text{ slug} = \frac{1\text{ lb}}{1\,\frac{\text{ft}}{\text{s}^2}}$$

- *Metric*

  10 milligrams (mg) = 1 centigram (cg)
  10 centigrams = 100 milligrams = 1 decigram (dg)
  10 decigrams = 100 centigrams = 1,000 milligrams = 1 gram (g)
  10 grams = 1 dekagram (dag)
  10 dekagrams = 1 hectogram (hg)
  10 hectograms = 1,000 grams = 1 kilogram (kg)

- *Conversion Between Systems*

  1 ounce ≈ 28.350 grams
  1 pound ≈ 453.592 grams
  1 U.S. ton (called a short ton) ≈ 0.907 metric ton

Note: These ounces are different than the fluid ounces listed under liquid capacity. These conversions assume weights as measured on the earth (as English units don’t usually measure true mass but just weight).

Time

60 seconds (s) = 1 minute (min)

60 minutes = 1 hour (hr)

24 hours = 1 day (d)

7 days = 1 week (wk)

365 days = 1 year (yr *or* y)

10 years = 1 decade

100 years = 10 decades = 1 century

1,000 years = 10 centuries = 1 millennium

Temperature

F = Temperature in Fahrenheit $(°\text{F}) = \frac{9}{5}C + 32$

C = Temperature in Celsius $(°\text{C}) = \frac{5}{9}(F - 32)$

Weight – U.S. Customary

16 ounces (oz) = 1 pound (lb)

2,000 pounds = 1 ton (called a short ton)

Other Units of Measure (*See Lesson 2.5 for a reminder on converting between some of these units.*)

- Electrical Charge

  C = **Coulombs**

  Charge of an electron = -1.602×10^{-19} C

- Electrical Resistance

  Ω = **Ohms** $\left(1\ \Omega = 1\frac{\text{V}}{\text{A}} = 1\frac{\text{J} \bullet \text{s}}{\text{C}^2}\right)$

- Electrical Voltage

  V = **Volt** $\left(1\ \text{V} = 1\frac{\text{J}}{\text{C}}\right)$

- Energy

  1 **Joule** (J) = $1\ \text{kg} \bullet \frac{\text{m}^2}{s^2}$

- Force

  Pounds (lb)

  1 **Newton** (N) = $1\ \text{kg} \bullet \frac{m}{s^2}$

  1 lb ≈ 4.448 N

- Frequency

  1 **Hertz** (Hz) = $1\ \text{s}^{-1}$

- Pressure

  Pa = **Pascal** $\left(1\frac{\text{N}}{\text{m}^2} = 1\ \text{kg} \bullet \frac{\text{m}}{\text{m}^2 \bullet \text{s}^2} = 1\frac{\text{kg}}{\text{m} \bullet \text{s}^2}\right)$

  1 **millimeter of mercury** (mm Hg) ≈ 133.322 Pa

- Power

  $W = \textbf{Watt}\left(1\text{ W} = 1\frac{\text{J}}{\text{s}} = 1\text{ kg} \bullet \frac{\text{m}^2}{s^3}\right)$

  1 **Horsepower** (hp) $= 745.7$ W

Other Reference

Prime Numbers Under 100

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47,
53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Fundamental Constants/Concepts

- ϕ Phi (the golden ratio) $= \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) \approx 1.61803398875$
- e (Euler's Number) ≈ 2.718281828459
- π Pi (the ratio of the circumference of a circle to its diameter) $\approx$ 3.14159265359 (To calculate, use your calculator's button or whatever rounded value you have memorized.)

Greek Alphabet

Α, α	Alpha	Η, η	Eta	Ν, ν	Nu	Τ, τ	Tau
Β, β	Beta	Θ, θ	Theta	Ξ, ξ	Xi	Υ, υ	Upsilon
Γ, γ	Gamma	Ι, ι	Iota	Ο, ο	Omicron	Φ, ϕ	Phi
Δ, δ	Delta	Κ, κ	Kappa	π, π	Pi	Χ, χ	Chi
Ε, ϵ	Epsilon	Λ, λ	Lambda	Ρ, ρ	Rho	Ψ, ψ	Psi
Ζ, ζ	Zeta	Μ, μ	Mu	Σ, σ	Sigma	Ω, ω	Omega

Symbols – Comparison

$\approx$ Approximately equals

$=$ Equals

$>$ Greater than

$<$ Less than

$\geq$ Greater than or equals

$\leq$ Less than or equals

Symbols – Sets

$\in$ Is an element of

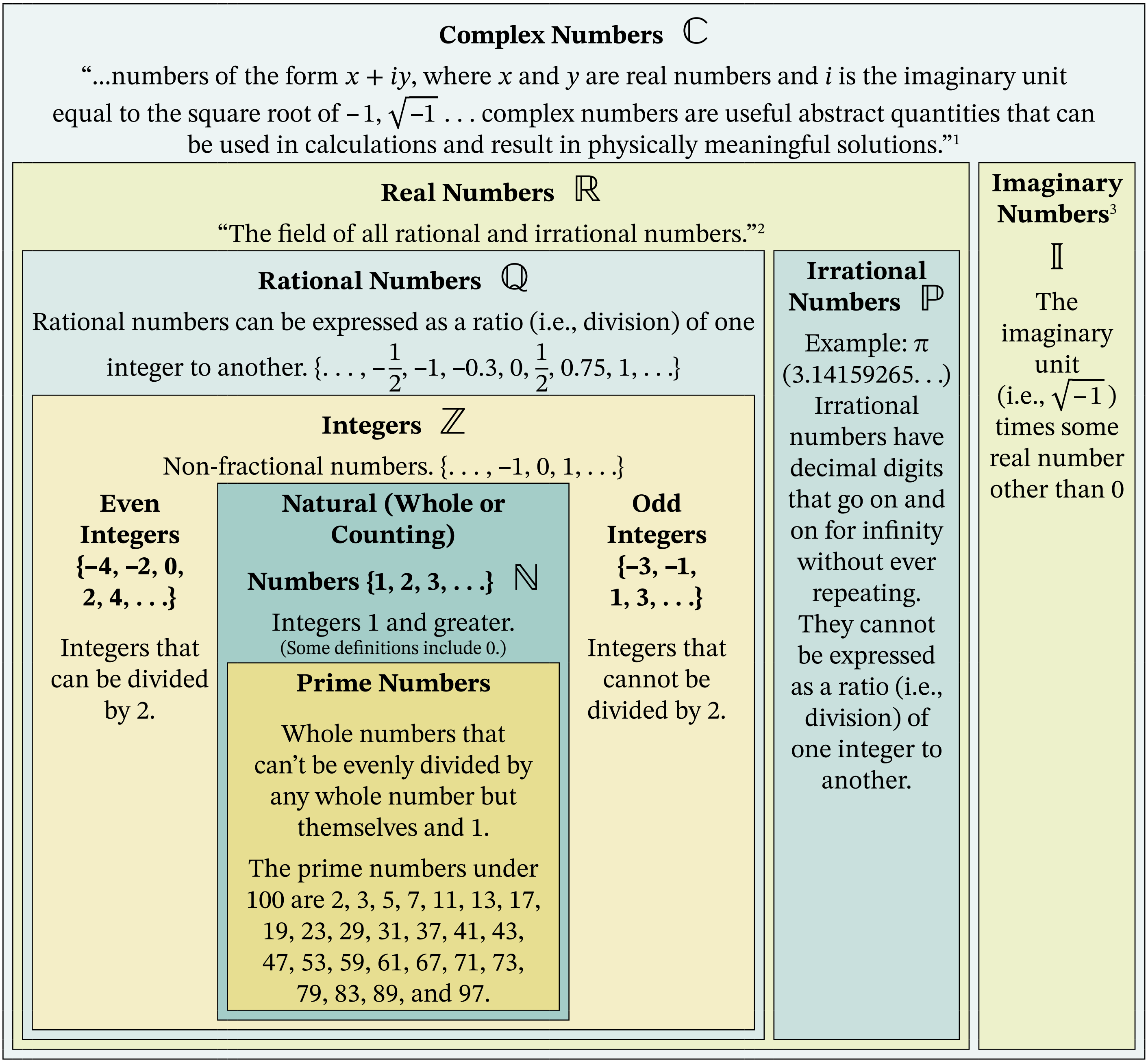
$\notin$ Is not an element of

$\subset$ Subset of

$\varnothing$ Empty set

$\cap$ Intersection of (think "AND")

$\cup$ Union of (think "OR")



Complex Numbers $\mathbb{C}$

"...numbers of the form $x + iy$, where x and y are real numbers and i is the imaginary unit equal to the square root of -1, $\sqrt{-1}$ . . . complex numbers are useful abstract quantities that can be used in calculations and result in physically meaningful solutions."[1]

Real Numbers $\mathbb{R}$

"The field of all rational and irrational numbers."[2]

Rational Numbers $\mathbb{Q}$

Rational numbers can be expressed as a ratio (i.e., division) of one integer to another. $\{. . . , -\frac{1}{2}, -1, -0.3, 0, \frac{1}{2}, 0.75, 1, . . .\}$

Integers $\mathbb{Z}$

Non-fractional numbers. $\{. . . , -1, 0, 1, . . .\}$

Even Integers $\{-4, -2, 0, 2, 4, . . .\}$

Integers that can be divided by 2.

Natural (Whole or Counting) Numbers $\{1, 2, 3, . . .\}$ $\mathbb{N}$

Integers 1 and greater.
(Some definitions include 0.)

Prime Numbers

Whole numbers that can't be evenly divided by any whole number but themselves and 1.

The prime numbers under 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

Odd Integers $\{-3, -1, 1, 3, . . .\}$

Integers that cannot be divided by 2.

Irrational Numbers $\mathbb{P}$

Example: π (3.14159265. . .) Irrational numbers have decimal digits that go on and on for infinity without ever repeating. They cannot be expressed as a ratio (i.e., division) of one integer to another.

Imaginary Numbers[3] $\mathbb{I}$

The imaginary unit (i.e., $\sqrt{-1}$) times some real number other than 0

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1 Complex number definition is from Eric Weisstein, "Complex Number," from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/ComplexNumber.html.

2 Real number definition is from Eric Weisstein, "Real Number," from *MathWorld*—A Wolfram Web Resource, http://mathworld.wolfram.com/RealNumber.html.

3 Imaginary Number definition is based on *Merriam-Webster.com Dictionary*, (Merriam-Webster), s.v. "pure imaginary," https://www.merriam-webster.com/dictionary/pure%20imaginary.