Chapter 1

An Introduction to Geometry

In this chapter, you will see how practical problems ranging from designing a city and measuring the earth to using shadows to tell time led to the development of geometry. The ideas that come from these problems are important because they lead to the solutions of other problems. As you proceed, you will become acquainted with geometric terms and ideas that will be useful as you continue your study of geometry.

City wall

Aqueduct

_Light-
house_

_Great
harbor_

_Pharos
town_

City wall

Lines in Designing a City

Alexandria, the city in which Euclid wrote his famous book on geometry, is named for Alexander the Great. Alexander had conquered much of the ancient world by the time of his death in 323 B.C. He is thought to have planned the streets of Alexandria, which today has become the second largest city in Egypt.
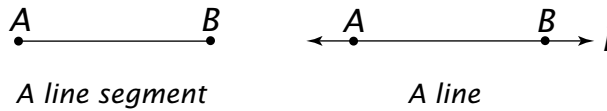
The map above shows the arrangement of the streets in ancient Alexandria. It also suggests some of the basic terms used in geometry. The streets lie along straight *lines;* they intersect in *points* and lie in a common *plane.*

Euclid described a ***point*** as "that which has no part." This description conveys the idea that, when we draw a dot on paper to represent a point, the point, unlike the dot, has a location only, without physical extent. The points in a figure to which we want to refer are labeled with capital letters.

Euclid described a ***line*** as having "breadthless length" and said that "the extremities of a line are points." These statements reveal that he was thinking of what we would now call a *line segment.*

•*A*

B•   •*C*

Some points

A ——————— B   A ←——————→ B _l_

A line segment   *A line*
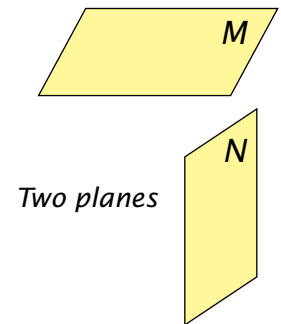
A **line segment** is part of a line bounded by two endpoints; it has a length that can be measured. Line segment AB in the figure at the left above has a length of 1 inch. A *line,* however, cannot be measured. The arrows on the figure of the line at the right above indicate that a line extends without end in both directions. These figures also suggest that lines, and hence line segments, are always *straight,* like a stretched string or the edge of a ruler.

Lines are usually named either by two points that they contain, such as line AB above, or by a single small letter. The line in the figure could be simply named line *l.*

The map of Alexandria is printed on a flat surface, which illustrates part of a **plane**. Planes are usually represented by figures such as the two at the right and are usually named, like points, with capital letters. These figures are useful in suggesting that planes are always *flat* but are unfortunately misleading in suggesting that planes have edges. In geometry, we think of a plane as having *no boundaries*. Although the parts of the planes shown here are bounded by edges, the complete planes extend beyond them, just as the line AB extends beyond the endpoints of the segment determined by A and B.
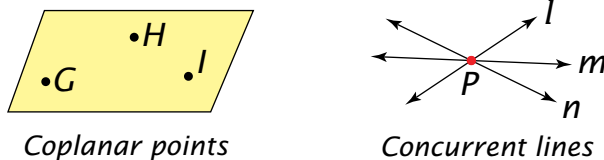
Some additional terms relating *points, lines,* and *planes* are illustrated and defined below.

M

N

Two planes

Definitions

A ——— B C →   E   F
                    D

Collinear points   *Noncollinear points*

Points are **collinear** if there is a line that contains all of them.
Points are **noncollinear** if no single line contains them all.

•H
•G   •I

Coplanar points

l
←——————→ *m*
P
n
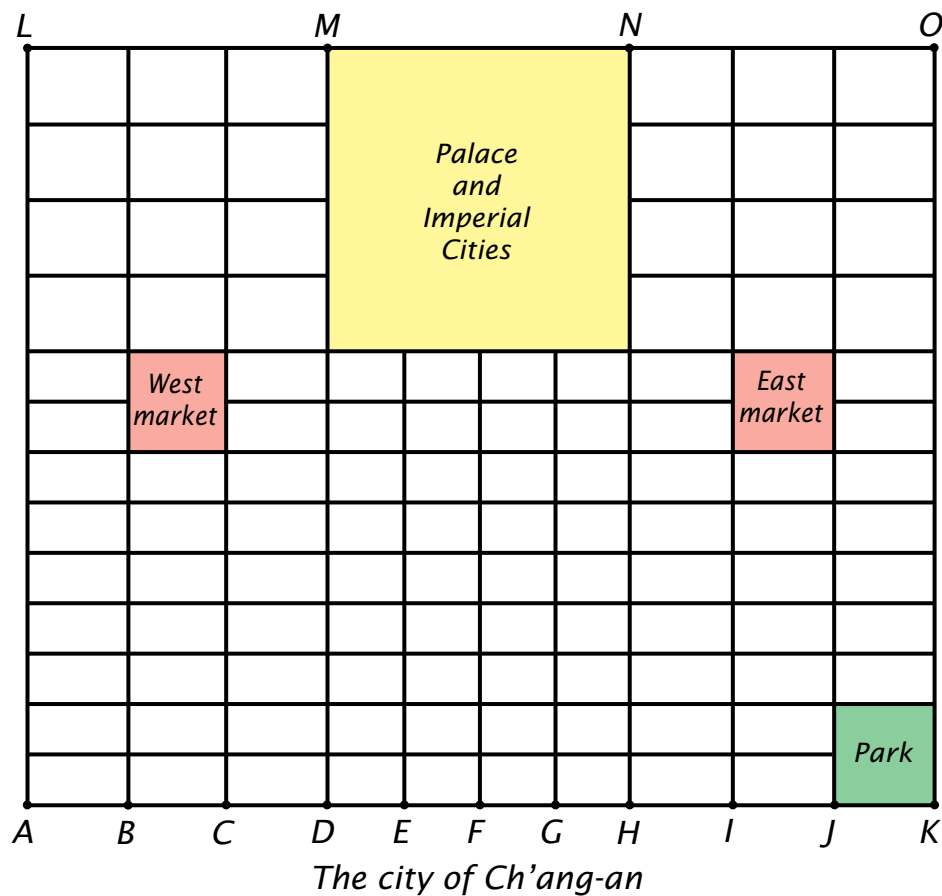
Concurrent lines

Points are **coplanar** if there is a plane that contains all of them.
Lines are **concurrent** if they contain the same point.

In the figure at the right above, lines *l, m,* and *n* all contain point P, their common *intersection.*

Exercises

Set I

This map shows the plan of the main streets of the ancient Chinese city Ch'ang-an.

The city of Ch'ang-an

The intersections of the street at the bottom of the map with the streets above it are labeled A through K.

1. What word is used to describe points such as these, given that one line contains them all?

Use your ruler to find the lengths of the following line segments in centimeters.

2. AK.
3. AF.
4. AD.
5. AB.
6. AL.

If the map is accurately drawn and the street from A to K is 6 miles long,

7. how many miles is it from A to F?
8. how many miles does 1 cm of the map represent?
9. how many miles is it from A to L?
10. Is the city square? Why or why not?
11. What regions of the city do appear to be square?

The city, whose name means "long security," was protected by a surrounding wall.

12. Use the lengths of the streets bordering the city to figure out how many miles long the wall was that surrounded it.

If the city were perfectly flat, then all of the points in it would lie in one plane.
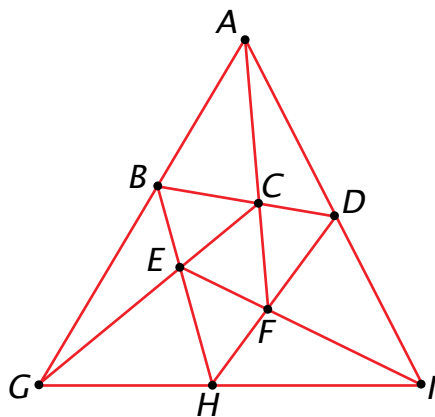
13. What word is used to describe points such as these?

14. Are the streets in the map represented by *lines* or *line segments?*

15. What is the difference between a line and a line segment?

Notice that AB, CD, and HK are three different segments, yet they all lie in *one* line.

16. On how many different lines do all the segments in this map lie?

Set II

The points and lines in the figure below are related in some unusual ways.

Each point in which line segments intersect has been labeled with a letter.

17. How many such points does the figure contain?

18. What does the word *collinear* mean?

The figure has been drawn so that each set of points that *appears* to be collinear actually *is.*

For example, points E, F, and I are collinear and lie on the line EI.

19. How many sets of three collinear points does the figure contain? List them, beginning with E-F-I.

20. What does the word *concurrent* mean?

How many different *lines* in the figure are concurrent at

21. point A?

22. point B?

23. point C?

24. each of the other points?

And now some very tricky questions about the line segments that can be named by using the letters in the figure.
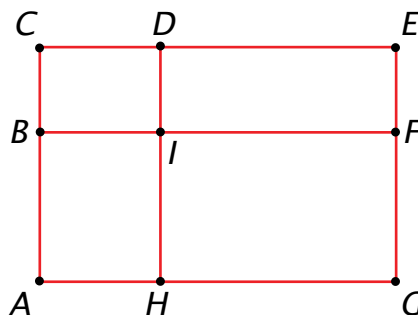
25. How many different line segments in the figure meet at point A? Name them.

How many different line segments meet at

26. point B? Name them.

27. point C? Name them.

Set III

On a sheet of graph paper, draw a figure similar to, but much larger than, the one below.

1. Use a colored pen or pencil to draw the following three lines:

AI, CF, and DG.

Draw enough of each line so that it crosses the entire figure.

2. What seems to be true about the three lines AI, CF, and DG?

Now use a pen or pencil of a different color to draw these three lines:

<div align="center">AF, CI, and EH.</div>

Extend each line to the edges of the figure.
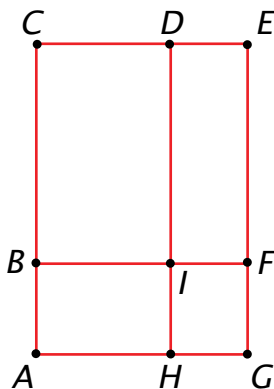
3. Describe what you see.

Now use a pen or pencil of a third color to draw these three lines:
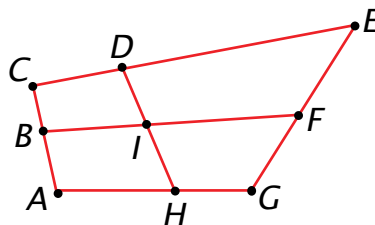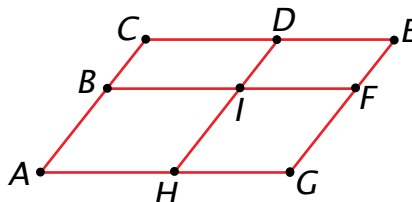
<div align="center">BG, CH, and EI.</div>

Extend each line to the edges of the figure.

4. What do you notice?

5. Draw the figure again, but change it so that it is taller or shorter, BF is higher or lower, and DH is farther to the left or right. An example of such a figure is shown here, but choose your own.

6. Now try drawing all the lines again and see what happens. What do you think?

7. Try changing the figure in other ways to see what happens to the lines drawn in the three colors. Some examples of altered figures are shown here. (Be sure to make your drawings much larger!)

8. What do you think?
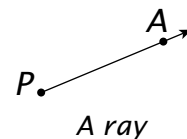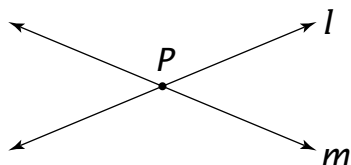
Angles in Measuring the Earth
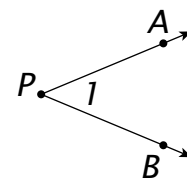
The ancient Greeks knew that the world was round. Eratosthenes, the head of the library in Alexandria where Euclid lived, figured out a clever way to estimate the distance around the world. His method depended on measuring an *angle.*

Euclid described an *angle* as "the inclination to one another of two lines in a plane which meet one another." If we draw a figure showing two lines, *l* and *m,* meeting at a point P, several angles are formed.
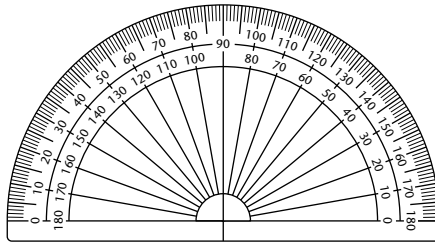
A ray

The figure can also be described as consisting of four *rays* starting from point P. As the figure suggests, a **ray** is part of a line that extends endlessly in one direction. To refer to a ray, we always name its endpoint first, followed by the name of any other point on it. The name of the ray in the first figure at the right is PA.

An **angle** is a pair of rays that have the same endpoint. The rays are called the *sides* of the angle and their common endpoint is called the *vertex* of the angle. Angles can be named in several ways. Using the symbol ∠ to mean "angle," we can name the angle in the second figure at the right ∠P or ∠1 or ∠APB or ∠BPA. Notice that, if the angle is named with three letters, the vertex is named in the middle.
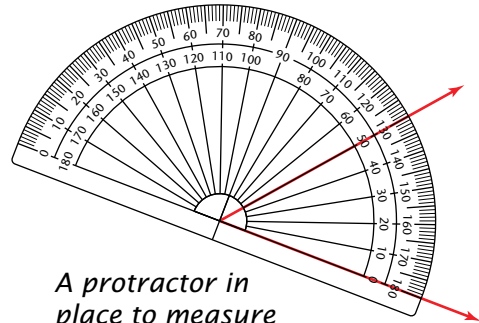
Two rays that form an angle

13

Before learning how Eratosthenes measured the earth, we will review the way in which angles are measured. To measure an angle, we need a *protractor*.
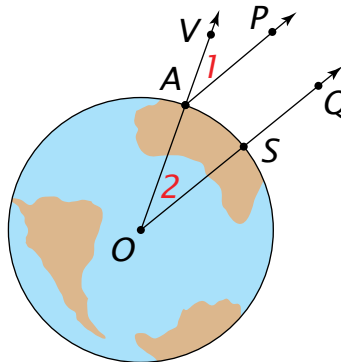
A protractor

A protractor in place to measure an angle

The protractor measures angles in *degrees* and has two scales, each numbered from 0 to 180. The center of the protractor is placed on the vertex of the angle so that the 0 on one of the scales falls on one side of the angle. The number that falls on the other side of the *same* scale gives the measurement of the angle in degrees. Look carefully at the scales on the protractor in the figure above to see why the angle has a measure of 50° and *not* 130°.

The diagram above illustrates the method that Eratosthenes used to measure the earth. He knew that Alexandria was about 500 miles north of the city of Syene (now called Aswan). The points A and S represent Alexandria and Syene, respectively, and the rays AP and SQ represent the direction of the sun as seen from each city. At noon on a certain day of the year, the sun was directly overhead in Syene, as shown by SQ. In Alexandria at the same time, the direction of the sun was along AP, in contrast with the overhead direction AV.
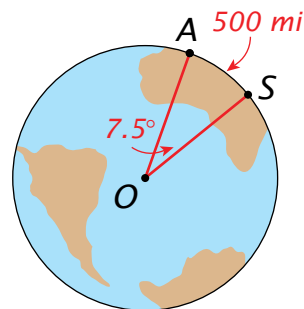
Eratosthenes measured the angle between these two directions, ∠1, and found that it was about 7.5°. Euclid had shown why it was reasonable to conclude that ∠1 was equal to ∠2, the angle with its vertex at the center of the earth. By dividing 7.5° into 360° (the measure of the degrees of a full circle), we get 48; so the full circle is 48 times the angle from Alexandria to Syene:

$$48 \times 7.5° = 360°.$$

Eratosthenes concluded that the earth's circumference (the entire distance around the circle) was 48 times the distance from Alexandria to Syene:
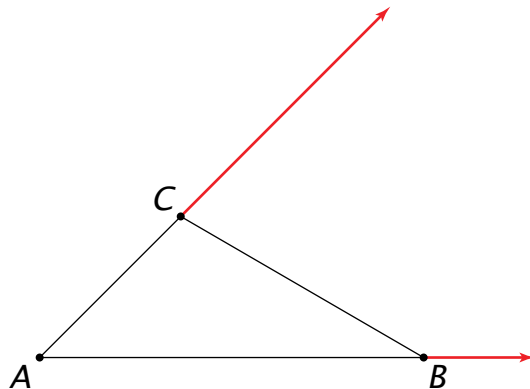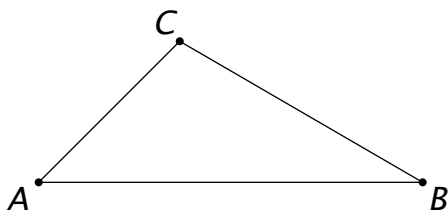
$$48 \times 500 \text{ miles} = 24{,}000 \text{ miles}.$$

The modern value of the circumference of the earth is about 25,000 miles; so Eratosthenes' estimate was remarkably accurate.

Exercises

Set I

Triangle Measurements. Each side and angle of the triangle above has a different measure. If you simply look at the figure without doing any measuring,

1. which side is the longest?
2. which angle is the largest?
3. which side is the shortest?
4. which angle is the smallest?

5. Use your ruler to measure the sides of the triangle, each to the nearest 0.1 cm.

Most protractors are too large to be able to measure the angles of a figure such as this triangle easily. For example, to measure ∠A, it is easiest to extend its sides as shown in color in the figure at the right above.

6. Name the two rays that are the sides of ∠A.
7. Use your protractor to measure ∠A.
8. Carefully trace triangle ABC on your paper and then extend the sides of ∠B.
9. Name the two rays that are the sides of ∠B.
10. Measure ∠B.
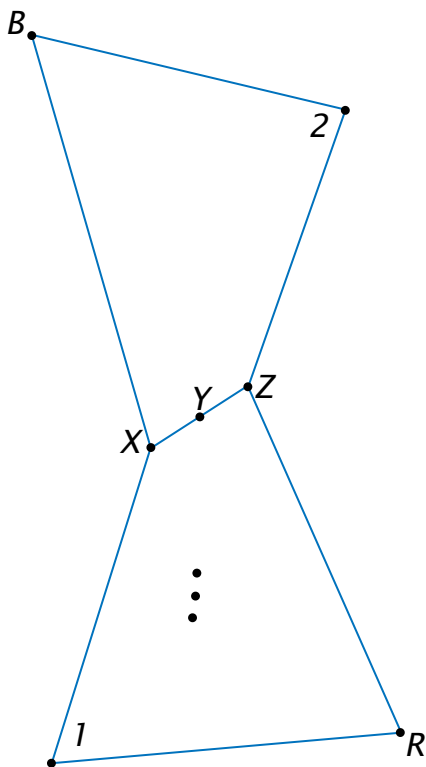
Go back to the figure you drew for exercise 8 and extend the sides of ∠C.

11. Name the two rays that are the sides of ∠C.
12. Measure ∠C.

Constellation Angles.

Eudoxus, a Greek mathematician, was the first to write about constellations, groupings of stars in the sky, in the fourth century B.C.

Two of the ten stars in the constellation Orion are among the brightest in the sky: Betelgeuse and Rigel. They appear as points B and R in the figure below.

13. What word describes the apparent relation of the points labeled X, Y, and Z in Orion's belt?
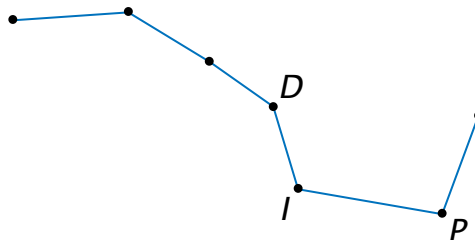
Look carefully at the four angles, ∠1, ∠B, ∠2, and ∠R. Without measuring, guess which of these four angles is

14. smallest.

15. largest.

Use a protractor to measure each of the four angles to check your guesses.

16. ∠1.   **18.** ∠2.

17. ∠B.   **19.** ∠R.

One of the most well-known constellations is Ursa Major, or the Big Dipper.
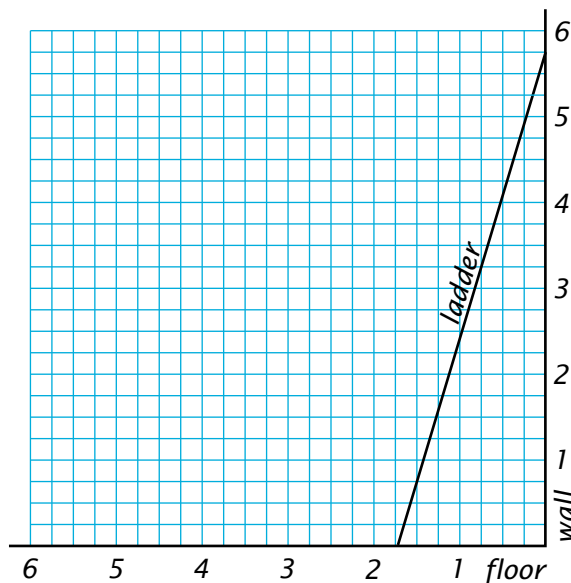
20. Carefully trace the figure above and then extend the sides of the angles as necessary to measure each of the following angles.

21. ∠D.

22. ∠I.

23. ∠P.

Set II

The Sliding Ladder. This figure represents a ladder 6 feet long leaning against a wall.

24. On a sheet of graph paper that is 4 units per inch, draw two lines to represent the wall and floor as shown in the figure. Mark a scale on the wall, letting 1 inch (4 units) represent 1 foot.

   The safest position in which to place the ladder is with its top $5^3/_4$ feet above the floor. Place the end of your ruler at the $5^3/_4$-foot point on the wall and turn the ruler until its 6-inch mark just touches the floor. Trace this safe position for the ladder on your paper.

25. Approximately how far is the foot of the ladder from the wall?

Use your protractor to measure the angle that the ladder makes

26. with the floor.

27. with the wall.

Now suppose that the ladder slips down the wall. Place the end of your ruler at the 5-foot point on the wall and turn it until its 6-inch mark touches the floor. Trace this position on your paper.

28. About how far is the foot of the ladder from the wall now?

Measure the angle that the ladder now makes

29. with the floor.

30. with the wall.

Place the end of your ruler at the 3-foot point on the wall, again being sure to turn it so that its 6-inch mark touches the floor.

31. About how far is the foot of the ladder from the wall now?

Measure the angle that the ladder now makes

32. with the floor.

33. with the wall.

Notice that, as the ladder's angle with the floor becomes smaller, its angle with the wall becomes larger.
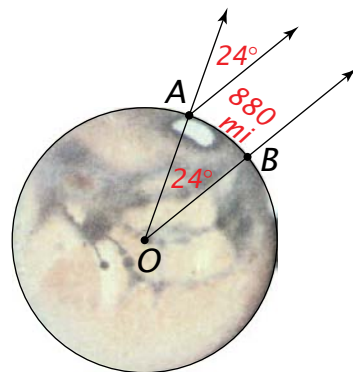
34. How big do you suppose each angle would be when the angles are equal?

Set III

Measuring Mars. The above map of the planet Mars was drawn in 1903 by the astronomer Percival Lowell. It shows canals and cities that he mistakenly thought had been built by intelligent beings on the planet.

   Suppose that when the sun is directly overhead at point B below, it is 24° from the vertical at point A.

   If the distance between the two locations is 880 miles, what is the distance around Mars? (Show your reasoning.)
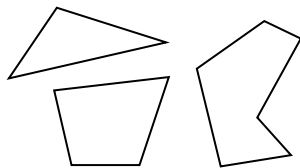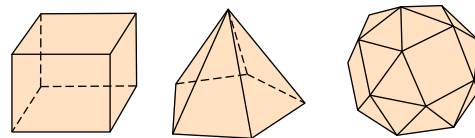
Polygons and Polyhedra: Pyramid Architecture

More than 2,000 years before Euclid was born, the ancient Egyptians used their knowledge of geometry to build the Great Pyramid at Giza, the only one of the "seven wonders of the world" still in existence. This pyramid is comparable in height to a building 40 stories high and covers an area of more than 13 acres. It was put together from more than 2 million stone blocks, weighing between 2 and 150 tons each!

Its base is almost a perfect square, and its four faces are in the shape of triangles. Squares and triangles are special types of *polygons,* and the pyramid itself is an example of a *polyhedron.* The words polygon and polyhedron are Greek in origin and, although the plural of polygon is polygons, the plural of polyhedron is *polyhedra.*
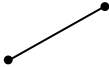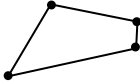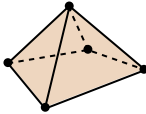
Polygons
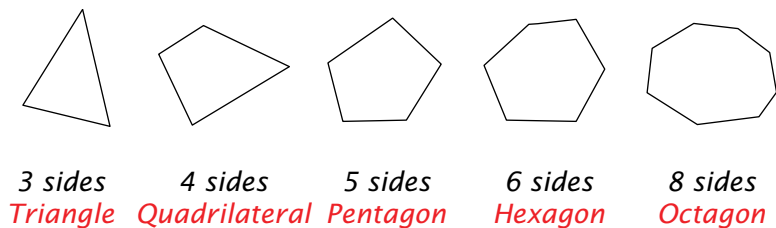
Polyhedra

Polygons and polyhedra are related to line segments in a simple way. As you know, a line segment is part of a line bounded by two endpoints; it is a one-dimensional figure. A *polygon* is bounded by line segments and lies in a plane; it is a two-dimensional figure. A *polyhedron* is bounded by polygons and exists in space; it is a three-dimensional figure.

Figure	*Dimensions*	*Example*
line segment	1	
polygon	2	
polyhedron	3	

The most basic way to name polygons is according to their number of sides. Some of them are illustrated here.

3 sides	4 sides	5 sides	6 sides	8 sides
Triangle	Quadrilateral	Pentagon	Hexagon	Octagon
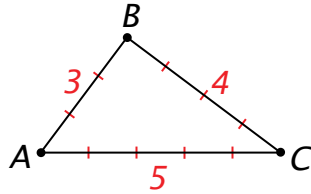
The ways for naming polyhedra are more complicated. Some words, such as *cube* and *pyramid,* are already familiar to you. Others, such as *tetrahedron* and *parallelepiped,* are probably not.

Whenever polygons or polyhedra are in practical problems, their measurements are often important. Consider the Great Pyramid, for example. A person seeing it for the first time might wonder how much ground it covers. In other words, what is the *area* of the polygon that is its base? Another question might concern how long it would take to walk around the pyramid. The answer depends on the *perimeter* of its base. A third question might concern how much stone was required to build the pyramid. This answer is related to the *volume* of the polyhedron that is its shape.

In this lesson, we will review two of these terms. The easiest one is perimeter. The *perimeter* of a polygon is the sum of the lengths of its sides.
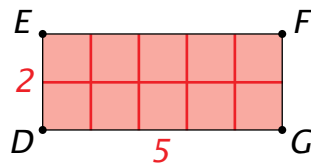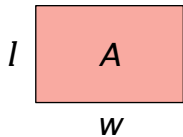
The perimeter of triangle ABC is $3 + 4 + 5 = 12$ units.

1 unit

1 square unit

The second term is *area*. According to one dictionary, area is "the surface included within a set of lines; specifically, the number of unit squares equal in measure to the surface."* This definition is obvious for a simple figure such as this rectangle.
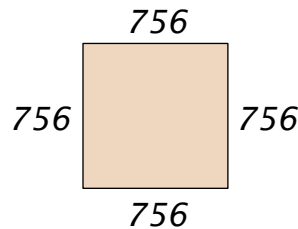
The area of rectangle DEFG is $2 \times 5 = 10$ square units.

The figure even suggests the simple formula for the area of any rectangle. A rectangle with length l and width w has area A given by

$$A = lw.$$

l A

w

The base of the Great Pyramid is square, with sides 756 feet long.

756

756   756

756

Its perimeter, then, is

$$756 \text{ feet} + 756 \text{ feet} + 756 \text{ feet} + 756 \text{ feet} =$$
$$4 \times 756 \text{ feet} =$$
$$3,024 \text{ feet}.$$

A square is also a rectangle; so its area is

$$756 \text{ feet} \times 756 \text{ feet} =$$
$$571,536 \text{ square feet}.$$

Merriam-Webster's Collegiate Dictionary.
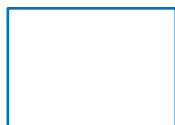
Exercises

Set I

The Perimeter and Area of a Square. The base of the Great Pyramid is in the shape of a square. A square is a rectangle all of whose sides are of equal length. Because a square is a *rectangle,* each of its angles is a *right angle;* that is, each angle has a measure of 90°.

Square *Rectangle*

1. Use your ruler and protractor to accurately draw a square whose sides are 3 inches long. Then divide it into smaller squares whose sides are 1 inch long.

2. What is the area of the square?

3. What is its perimeter?

Without drawing it, imagine a square whose sides are 6 inches long.

4. What is its area?

5. What is its perimeter?

Without drawing it, imagine a square whose sides are 1 foot long.

6. What is its area in square feet?

7. What is its perimeter in feet?
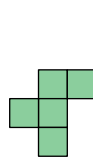
Now think of the same square as having sides 12 inches long.

8. What is its area in square inches?

9. What is its perimeter in inches?

If a square has sides that are x units long, what, in terms of x, is its

10. area?

11. perimeter?

Pentominoes. A game called Pentominoes uses the 12 pieces shown here. Their names are suggested by their shapes.
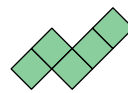
F *I* *L* *N*

P *T* *U* *V*

W *X* *Y* *Z*

12. Each piece is in the shape of a *polygon.* What is a polygon?

13. Which piece is in the shape of a *quadrilateral?* Why?

14. All 12 pieces have the same area. What is it?

15. All 12 pieces have the same perimeter *except for one.* Which one is it?

Texas Ranches. Two ranches are for sale in Texas. Each is in the shape of a rectangle and their dimensions are as follows:

Ranch A: 4 miles wide, 14 miles long.
Ranch B: 6 miles wide, 10 miles long.

16. Suppose that neither ranch has a fence. Which one would need more fencing to surround it?

17. Both ranches are the same price and you want to buy the bigger one. Which one should you buy?

18. Which is more important in this situation: *perimeter* or *area?*
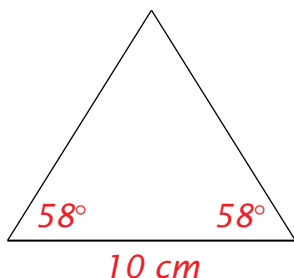
Set II

A Model of the Great Pyramid. While on a trip to study the Great Pyramid in 1799, Napoleon made this sketch together with some notes.

Use your ruler and protractor to draw a square 10 cm on each side for the base of the pyramid on a heavy sheet of paper.

The other four faces of the pyramid are in the shape of triangles. Use the measurements shown in the figure below to draw four triangles on a sheet of heavy paper.

58°   58°
10 cm

19. Use your protractor to measure the angle at the top of one of these triangles.

20. Do any sides of the triangle appear to be equal? Use your ruler to check your answer.

21. Cut out the square and four triangles and tape them together to make a scale model of the Great Pyramid.

22. Make a drawing of the pyramid.

The pyramid has five faces altogether: its square base and its four triangular faces.
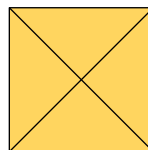
23. What kind of figures are squares and triangles?

24. What kind of figure is the pyramid itself?

The line segments in which the faces of the pyramid meet are called its *edges.*

25. How many edges does the pyramid have?

26. How many corners does it have?

27. How many edges meet at the top of the pyramid?

28. How many edges meet at each corner of the base?

Each face of the pyramid lies in a *plane.*

29. In how many different planes do the faces of the pyramid lie?

30. If two *lines* intersect, they intersect in a *point.* If two *planes* intersect, in what do they intersect?

31. Use your ruler to estimate, as best as you can, the height of your model in centimeters.
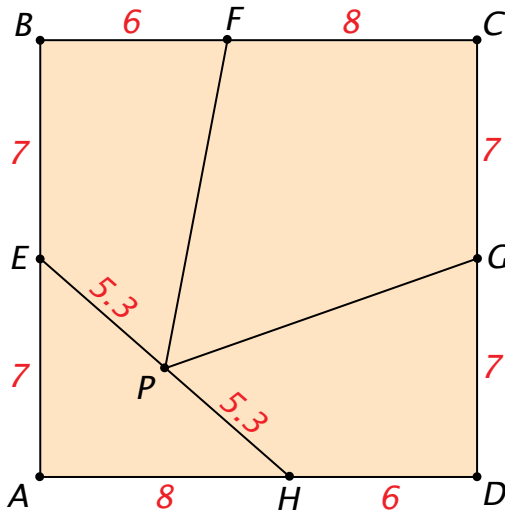
In your model, 10 cm represent 756 feet, the length of one edge of the base of the pyramid.

32. How many feet does 1 cm represent?

33. Use this result and your estimate of the height of your model in centimeters to estimate the height of the Great Pyramid in feet.

34. What view of the Great Pyramid does this drawing appear to show?

Set III

A Square Puzzle. The book *Mathematical Snapshots* by the Polish mathematician Hugo Steinhaus begins with an illustration of four polygons arranged to form a square.[*]

Use your ruler and protractor to make an enlargement of this figure on a sheet of heavy paper by doing each of the following:

(1) Draw a square 14 cm on each side and label its corners A, B, C, and D as shown in the figure.

(2) Mark the points E, F, G, and H at the indicated distances from the corners.

(3) Draw EH.

(4) Mark point P on EH.

(5) Draw PF and PG.

Test the accuracy of your drawing and your ability to measure angles by doing each of the following.

1. ∠EPF, ∠FPG, and ∠GPH should appear to be equal. What is the measure of each of these angles? Write the measure inside each angle.

2. ∠BFP should appear to be 30° larger than ∠AEH. What is the measure of each of these angles?

3. ∠BEH should appear to be 90° larger than ∠EHA. What is the measure of each of these angles?

4. Two other line segments in the figure should appear to have the same length as that of EH. Which are they?

5. Carefully cut the four polygons out. Try to rearrange them to form an equilateral triangle (a triangle all of whose sides are equal). Trace the arrangement on your paper.

6. About how many centimeters long is each side of the equilateral triangle?

7. How large is each of its three angles?

8. How does the area of the triangle compare with that of the original square? Why?

9. What is the area of the triangle?

10. How does the perimeter of the triangle compare with that of the original square?

---

[*]The figure is the solution to a puzzle created in 1902 by Henry Dudeney, England's greatest inventor of mathematical puzzles. The lengths in it have been rounded for convenience.

Constructions: Telling Time with Shadows

The largest sundial in the world, built in 1724 in Jaipur, India, is shown in the photograph above. Its gnomon (the object that casts the shadow) is a ramp that rises 90 feet above the ground. The shadow cast by the ramp moves at a rate of about two inches per minute, making it possible to accurately tell the local time within seconds!

Sundials were the main instrument for keeping time from the days of ancient Egypt until the sixteenth century. At noon, the sun is at its highest in the sky, the shadow cast by the gnomon is at its shortest, and, in the northern hemisphere, the shadow points northward.

The Egyptians were aware of these facts and put them to use in constructing their pyramids so that their sides faced directly north, south, east, and west. The method that they are thought to have used for finding map directions is related to a geometric construction.
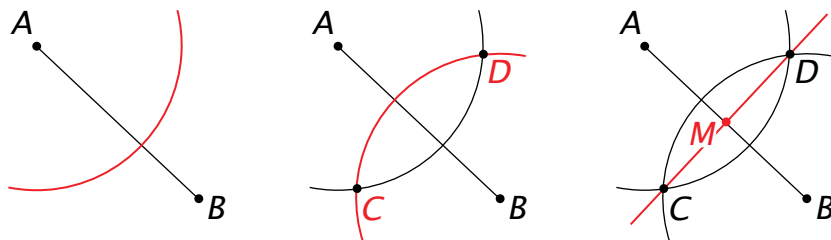
Since the time of Euclid, two tools have been used in making geometric drawings: the ***straightedge*** and ***compass***. The straightedge is used for drawing *lines* (actually *line segments*), and the compass is used for drawing *circles* or parts of circles called *arcs*. A ***circle*** is the set of all points in a plane that are a fixed distance from a given point in the plane. The fixed distance is called the ***radius*** of the circle and the given point is called its *center*.

Drawings made with just a straightedge and compass are called ***constructions*** to distinguish them from those made with other tools such as a ruler and protractor.

In this lesson, we will learn simple constructions for bisecting line segments and angles. To *bisect* something means to divide it into two equal parts.

Construction 1
To bisect a line segment.
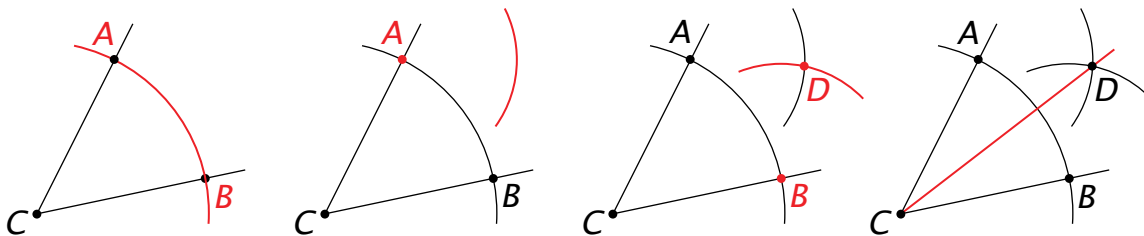
Let AB be the given line segment. With A as center, draw an arc intersecting the segment as shown in the first figure. With B as center, draw an arc of the same radius that intersects the first arc in two points (C and D in the second figure). Draw line CD. Line CD bisects segment AB. In other words, AM and MB have the same length.

There is a simple way to prove that this method does what we have claimed. You will learn how it works later in your study of geometry.

Angles can also be bisected with a straightedge and compass.

Construction 2
To bisect an angle.
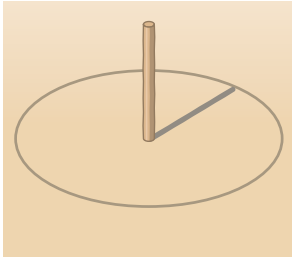
Let ∠C be the given angle. With point C as center, draw an arc that intersects the sides of the angle (at points A and B in the figure). With A as center, draw an arc inside the angle as shown in the second figure. With B as center, draw an arc of the same radius that intersects the arc drawn from point A (at point D in the third figure). Draw line CD. It bisects ∠C. In other words, ∠ACD and ∠DCB are equal in measure.
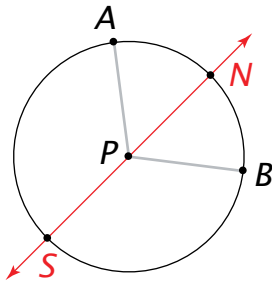
This equality also can be easily proved, as you will learn later.

Exercises

Set I

Finding North by Shadows. The ancient Egyptians are thought to have used the shadow of a vertical post to find north. The figure below is an overhead view of some level ground into which a post has been stuck at point P. Once in the morning the tip of the post's shadow touches the circle at A, and once in the afternoon it touches the circle at B. These two positions of the shadow, PA and PB, form ∠P.
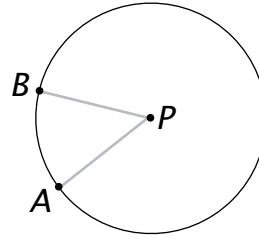
1. Draw a circle larger than this one, label its center P, mark two points on the circle A and B, and draw PA and PB. Bisect ∠P.

   The ray that bisects ∠P points north. Label the point in which it intersects the circle N. Also draw the ray from P that points south.

2. Check the accuracy of your drawing by using your protractor to measure ∠APB and ∠APN. What measurements do you get?

3. Apply the directions given for exercise 1 to the following figure.

4. Check the accuracy of your drawing by using your protractor to measure ∠APB and ∠APN.
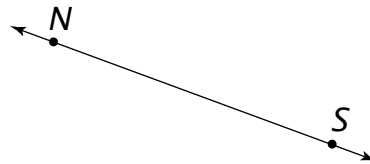
   In each of your drawings, points P, A, and B do not lie on the same line.

5. What word describes such points?

   On the other hand, points P, A, and B do lie in the same *plane*.
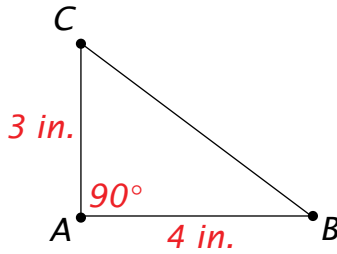
6. What word describes these points?

   Suppose line NS in the figure below points north and south.

7. Draw a figure like this one on your paper and then use your straightedge and compass to construct the line that bisects segment NS. In what directions do you think this line points? Label them on the line.

8. Use your protractor to measure the angles formed by NS and EW. Do you get the results that you would expect?

9. What is the sum of all four angles? What do you expect it to be?

10. Use your ruler and protractor to draw a triangle with the measurements shown in the figure above. Then use your straightedge and compass to bisect each side of the triangle.

11. What relation do the three bisecting lines that you drew seem to have to each other?

12. Where do the lines seem to meet?

   Label the point in which they meet P and draw PA.
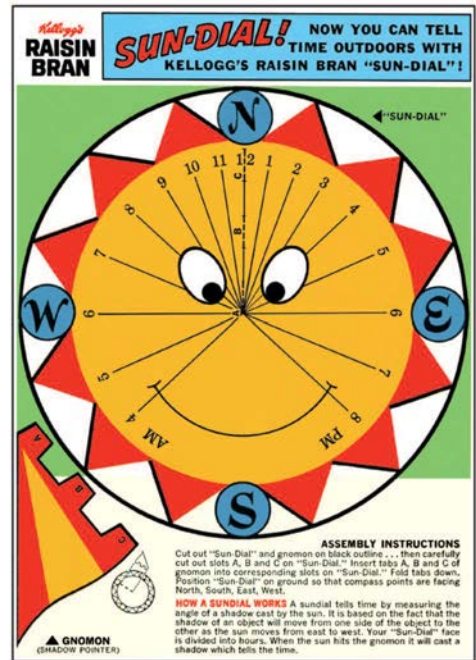
13. How do the distances from point P to the corners of the triangle, A, B, and C, seem to compare?

14. Make another copy of the same triangle. This time, use your straightedge and compass to bisect its angles.

15. What seems to be true about the three lines that you drew?

   Label the point in which they meet S.

16. Are the distances from point S to the corners of the triangle equal?
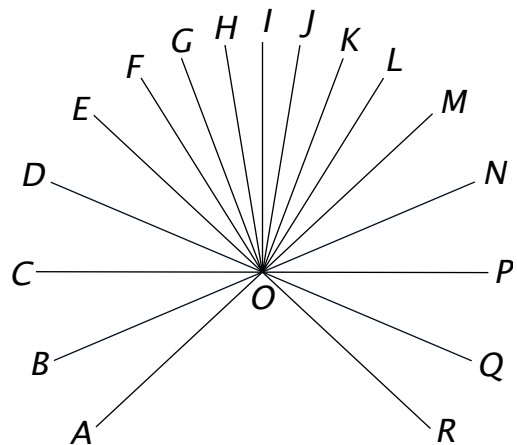
Set II
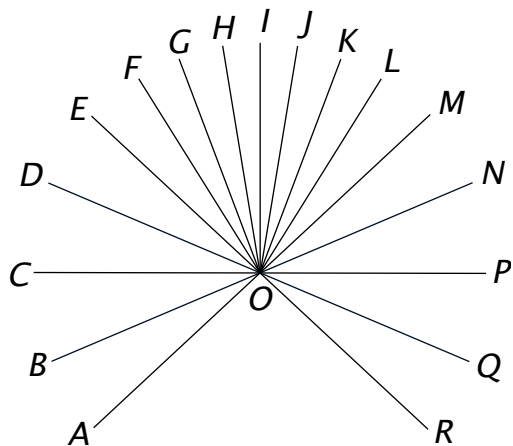
About Sundials. A sundial was once featured on boxes of Kellogg's Raisin Bran. When the gnomon (shadow pointer) was cut out and put in place, its shadow moved around the dial during the day. The way in which the hour lines are spaced suggests that the shadow does not always move at the same speed.

In what map direction does the shadow

17. point at 6 o'clock in the morning?

18. point at noon?

19. never point?

20. At what time of the day do you think the shadow moves the most slowly?

21. When does it appear to move fastest?

   The hour lines of the Kellogg sundial have been redrawn and lettered in the figure below.
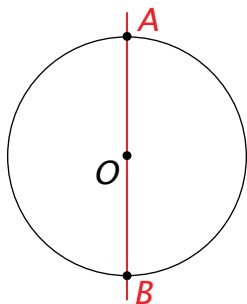
22. Which angle in the figure seems to be equal to ∠HOI?

23. If your answer to exercise 22 is true, what does OI do to ∠HOJ?

24. Does OQ seem to bisect ∠POR? Why or why not?

25. Name two angles in the figure that OC seems to bisect.

26. Which angle seems to be equal to ∠COF?

27. Name three angles in the figure that seem to be equal to ∠COD.

28. Which angle seems to be equal to ∠AOR?
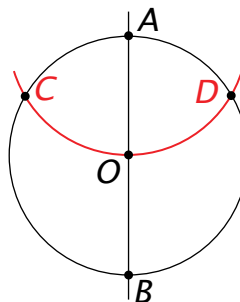
A sundial in which the hour lines are evenly spaced is simpler to make but harder to use. A straightedge and compass can be used to construct such a sundial without the use of a protractor.

29. Use your compass to draw a circle with a radius of 3 inches. Label its center O.
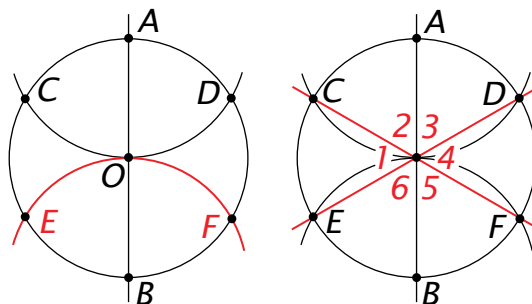
Draw a line through O as shown in the figure at the bottom of the preceding column and label the points in which it intersects the circle A and B.

With your compass, draw an arc centered at A and passing through O as shown in the figure below. Label the points in which the arc intersects the circle C and D.

Draw another arc centered at B and passing through O as shown in the left-hand figure below. Label the points in which the arc intersects the circle E and F.
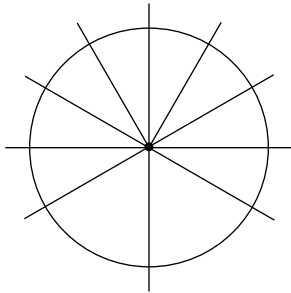
If you have drawn your figure accurately, points C, O, and F should appear to be collinear, as should points E, O, and D.

30. What does collinear mean?

Draw the lines CF and ED and number the angles at the center as shown in the right-hand figure above. Each of the six numbered angles in your drawing should appear to be equal. Measure them with your protractor.
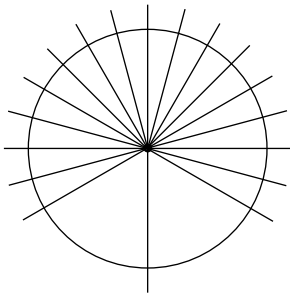
31. What does the measure of each angle appear to be?

Use your straightedge and compass to bisect ∠1, ∠2, ∠3, and ∠4. The lines in your drawing should now look like this.

32. What is the measure of each of the eight smaller angles in the figure?

Use your straightedge and compass to bisect each of these angles. The lines in your drawing should now look like this.

Compare your drawing with the Kellogg's sundial and number the hour spokes appropriately.

If your sundial had a gnomon perpendicular to its base and if the sundial were tilted in the right direction, the shadow would move at a steady speed across the hour lines.
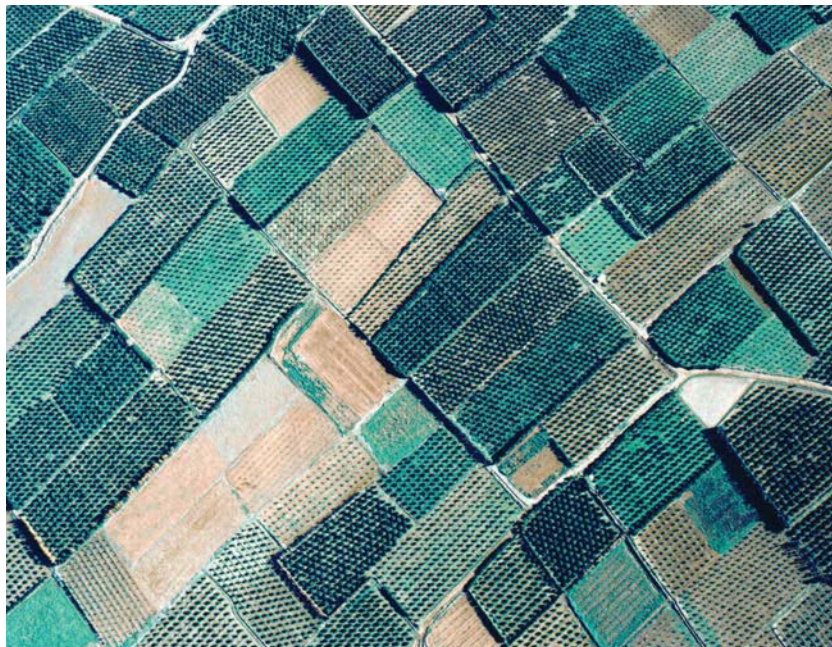
Through how many degrees would the shadow turn

33. between 6 A.M. and 6 P.M.?

34. between noon and 6 P.M.?

35. during one hour?

36. How long would it take the shadow to turn 1 degree?

Set III

Angle Bisectors in a Rectangle. Something rather surprising usually happens when the angles of a rectangle are bisected. The following exercises will help you discover what it is.

1. On a sheet of graph paper ruled 4 units per inch, draw a rectangle having sides with lengths 1.5 and 2 inches.

2. Draw the four lines that bisect the angles of the rectangle. What polygon do they seem to form?

3. Draw another rectangle having sides of lengths 2 inches and 3 inches. Draw the lines that bisect its angles. What do you notice?

4. Follow the directions given in exercise 3 but with a rectangle of sides 2 inches and 4 inches. What do you notice?

5. Follow the directions given in exercise 3 but with a rectangle of sides 1 inch and 4 inches. What do you notice?

6. Is the size of the polygon formed related in any way to the shape of the rectangle? Explain.

7. Do you think there is any rectangle for which the polygon would shrink to a point? Explain.

An overhead view of plots of land
in the shape of quadrilaterals not
far from Alexandria.

We Can't Go On Like This

This title may seem to be a very strange one for a geometry lesson, but it points out a troubling problem with what we have been doing so far.

We have become acquainted with many important ideas in geometry, mostly in an informal way. We have learned how to make simple constructions with a straightedge and compass, yet we don't really know how they work. We have guessed some conclusions on the basis of the appearance of things, not always being certain that they were true. Our approach to geometry to this point has been comparable to that of the ancient Egyptians, long before the time of Euclid.

What has survived of Egyptian mathematics as it existed 4,000 years ago reveals an accumulation of rules and results seemingly based on trial and error. The Egyptians were evidently satisfied with ideas that appeared to be useful, without concerning themselves about why they might or might not always be true.
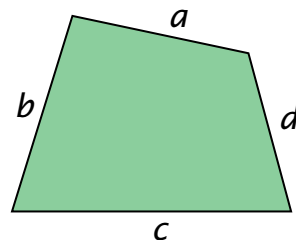
Here is an example of the difficulties to which this approach to geometry leads. To find the areas of fields in the shape of quadrilaterals, the Egyptians used the formula

$$A = \frac{1}{4}(a + c)(b + d)$$

in which *a, b, c,* and *d* are the lengths of the consecutive sides. They had gotten this formula from the ancient Babylonians. The trouble with it is that sometimes it gives the correct answer and sometimes it does not.

Thus when tax assessors used it to figure out the areas of four-sided fields, some landowners were charged correctly and some were not. In fact, those who were not were cheated every time! The tax assessors may have known that these charges were incorrect, but apparently no one could figure out what was wrong with the formula to cause it to give such undependable results.

We will explore this problem and some other similarly troubling ones in this lesson.

Exercises

Set I

The Egyptian Tax Assessor. We are back in 2500 B.C. and three Egyptians, Ramses, Cheops, and Ptahotep, have recently purchased plots of land in the shapes shown here.

The tax assessor uses his formula for the area of a quadrilateral,

$$A = \frac{1}{4}(a + c)(b + d)$$
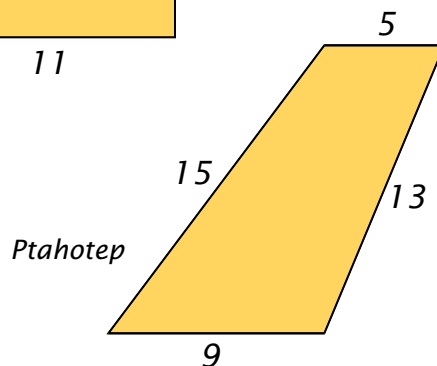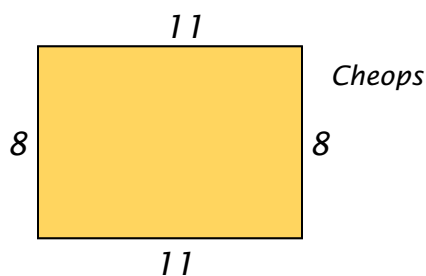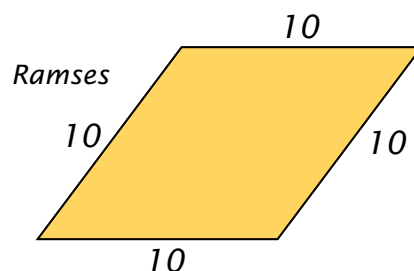
in which *a, b, c,* and *d* are the lengths of the consecutive sides.

What area does he calculate for the plot belonging to
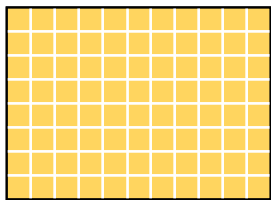
1. Ramses?

2. Cheops?

3. Ptahotep?

The bigger the plot, the higher the tax.

4. According to these results, who has to pay the highest tax?
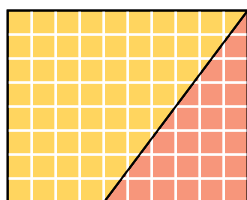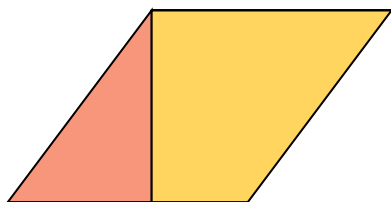
The area of Cheops's plot, being rectangular in shape, is easiest to check.

5. Does it contain as many square units as the tax assessor says?

Before we try to count the number of squares in each of the other plots, making some changes in the figures will make the job easier.
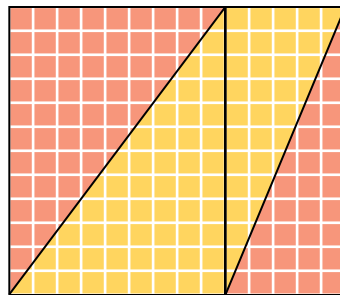
For Ramses' plot, we can imagine cutting off the triangular part on the left and moving it to the right.

6. How many square units does the plot contain?

For Ptahotep's plot, we can add some lines as shown at the top of the next column.

7. How many square units does the left-hand rectangle contain?

8. How many square units do you think Ptahotep's part of this rectangle contains?

9. How many square units does the right-hand rectangle contain?

10. How many square units do you think Ptahotep's part of this rectangle contains?

11. Use these results to determine how many square units Ptahotep's property contains.

12. Which one of the three is cheated the most by the tax assessor's formula? Why?
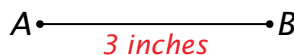
How to calculate area correctly is an important topic in geometry. Having worked through these exercises, you will probably agree that it would be convenient to have easier methods whose logic we really understand.

Set II
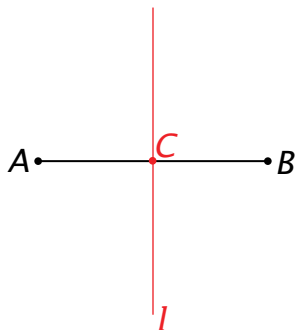
Trisection Problems. The methods for bisecting line segments and angles by using just a straightedge and compass were known to the ancient Greeks long before Euclid was born. Having solved these two problems, the Greeks naturally wondered if just these two tools could be used to *trisect* a line segment or angle, that is, divide them into *three* equal parts.

The trisection problem for line segments was solved almost immediately. Here is one way to do it.
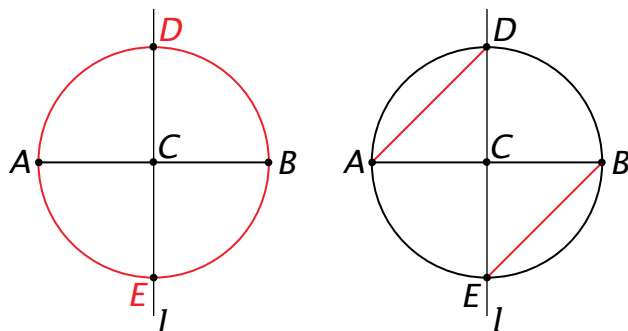
13. For convenience in checking the result at the end, draw a line segment AB 3 inches long.

A •————————————• B
3 inches

First, use your straightedge and compass to construct the line, *l*, that bisects AB. Label the midpoint C.
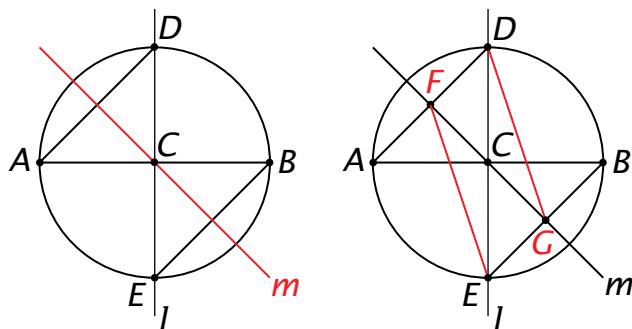
With C as center, draw a circle through A and B. Label the points in which line *l* intersects the circle D and E. Draw AD and BE.
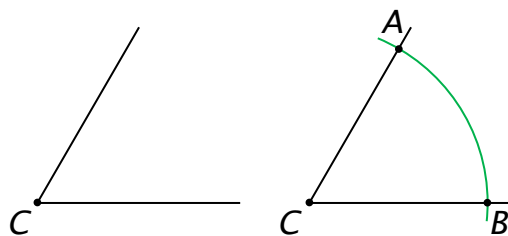
Construct the line that bisects AD and label it *m*. (You should notice that line *m* goes through C, the center of the circle.)

Where line *m* intersects AD and BE, label the points F and G as shown here. Draw DG and FE.
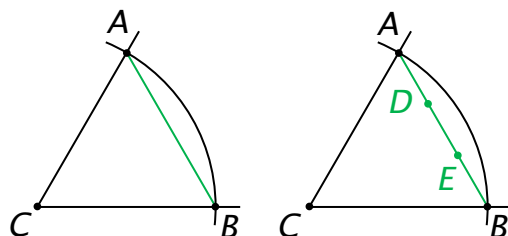
These two line segments trisect the original line segment AB, which you can check with your ruler.

An early method proposed for trisecting angles is illustrated below.
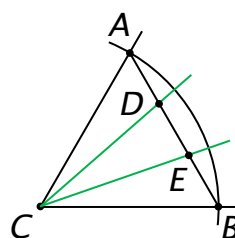
Start with angle C   *Draw an arc centered at C*

Draw AB   *Trisect AB*

Draw CD and CE

Rather than carrying out this construction with straightedge and compass, we will draw an accurate figure to test the results.

14. Use your ruler and compass to draw an equilateral triangle, each of whose sides is 3 inches long. Label its vertices A, B, and C to correspond to the third figure above. Use your ruler to find the points D and E on side AB that trisect it. Draw CD and CE.

15. Center your protractor on C and carefully measure ∠ACD, ∠DCE, and ∠ECB. What does the measure of each angle seem to be?
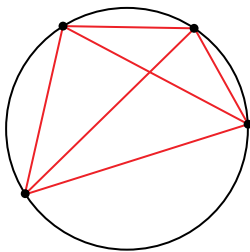
Because we can neither draw nor measure things perfectly, it is not easy to tell whether the original angle has been trisected. In fact, it *has not.* As surprising as it may seem, since the time of Euclid, no one has ever found a way that can be used to trisect any angle by using just a straightedge and compass!

We have made two drawings, both of which might *seem* to be correct, yet one is not. The title of this lesson, "We Can't Go On Like This," describes our dilemma. It is not possible to be sure of what we are doing in geometry without developing a deeper understanding of the subject.

Set III

Cutting Up a Circle. Here is a simple problem about points and circles that has become famous because it is so baffling.

If four points are chosen on a circle and connected with line segments in every possible way, the result might look something like this. Notice that the circle is separated into eight regions.

1. Draw a circle, choose five points on it, and connect them with line segments in every possible way.

2. Into how many regions is the circle separated?

How many regions would be formed if you started with

3. just two points?

4. three points?

5. Put all of these results together by making a table like this:

Number of points connected: 2 3 4 5
Number of regions formed: ? ? 8 ?

6. What happens to the number of regions each time one more point is added to the circle?

How many regions would you expect to be formed if you started with

7. six points?

8. seven points?

9. Make drawings of circles having six and seven points. Start with large circles.

10. Are the results what you expected?

This engraving from 1630 is titled *Dido Purchases Land for the Foundation of Carthage.* It depicts the hide of an ox being cut to solve a problem in geometry considered in this review.

Summary and Review

Introduction (pp. 1–2)

Since about 300 B.C., Euclid's book titled the *Elements* has been the model for presenting geometry in a logical way. We will become acquainted with Euclid's approach to mathematics in our study of geometry.

Lesson 1 (pp. 8–9)

Some basic terms used in geometry are *point, line,* and *plane.* A *line segment* is part of a line bounded by two endpoints.

Some terms relating points, lines, and planes are *collinear* (points that are on the same line), *noncollinear* (points that are not on the same line), *coplanar* (points that are in the same plane), and *concurrent* (lines that contain the same point).

Lesson 2 (pp. 13–15)

A *ray* is part of a line that extends endlessly in one direction. An *angle* is a pair of rays (called its *sides*) that have the same endpoint (called its *vertex*).

A *protractor* measures angles in *degrees.*

Lesson 3 (pp. 18–20)

A *polygon* is bounded by line segments and lies in a plane. A *polyhedron* (plural *polyhedra*) is bounded by polygons and exists in space.

Names for some common polygons include: *triangle* (three sides), *quadrilateral* (four sides), *pentagon* (five sides), *hexagon* (six sides), and *octagon* (eight sides).

Two numbers associated with a polygon are its *perimeter* (the sum of the lengths of its sides)

and its *area* (the number of unit squares equal in measure to its surface).

A rectangle with length l and width w has area A given by

$$A = lw.$$

Lesson 4 *(pp. 24–25)*

Geometric drawings called *constructions* are made with just two tools: the *straightedge* (for drawing lines) and the *compass* (for drawing circles).

Two simple constructions are to *bisect a line segment* and to *bisect an angle.* To bisect something means to divide it into two equal parts.

Lesson 5 *(pp. 30–31)*

The study of geometry in an informal way, as we have done in this chapter, can lead to many interesting and practical results. This informal approach is ultimately unsatisfactory, however, because conclusions based on appearances or guessing may not always be correct.

Exercises

Set I
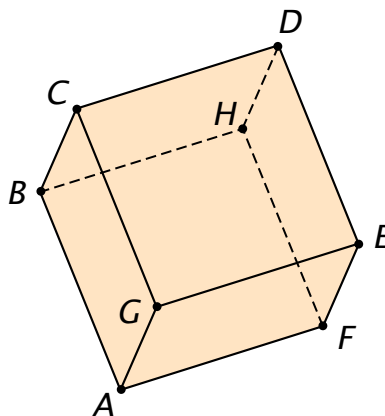
The die shown in the photograph above was made in Italy in about 700 B.C. and is in the shape of a cube.

The corners of a cube are *points.* What kind of geometric figure

1. are its edges?

2. are its faces?

3. is the cube itself?

The figure below is a transparent view of a cube.

How many edges

4. meet at each corner of a cube?

5. does a cube have altogether?
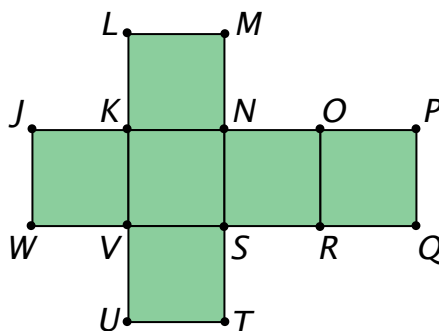
How many faces

6. meet at each corner of a cube?

7. does a cube have altogether?

The corners A, G, and C of the cube are noncollinear.

8. What is another word that describes them?

The following pattern can be used to form a cube.

9. In how many lines do the edges of the pattern lie?

Some harder questions

Suppose the pattern is cut out and folded together to form a cube.
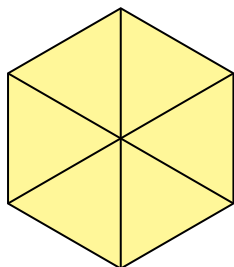
With which edge would each of the following edges come together?

10. JW.

11. MN.

12. LM.

With which corner(s) would each of the following corners come together?

13. R.

14. W.

Here is another transparent view of a cube.
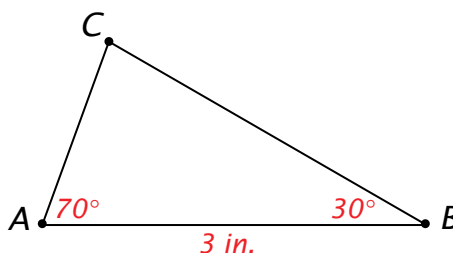
15. Looking at this figure as flat rather than three-dimensional, what polygons do you see in it?

16. Make a copy of the figure, but make the figure look three-dimensional by drawing it with the hidden edges as dashed line segments.

17. What is the greatest number of faces of a die that can be seen at any one time?

Set II
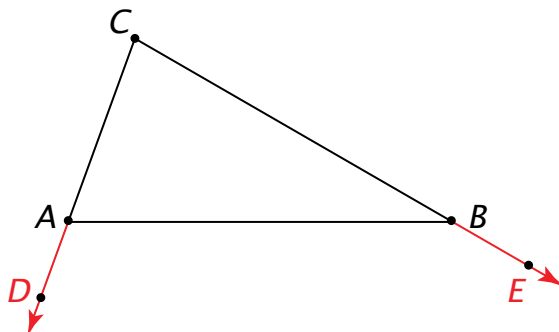
Some drawings in geometry are made with a ruler and protractor, and others are made with just a straightedge and compass.

18. What are drawings made with just a straightedge and compass called?

19. Use your ruler and protractor to draw a

triangle with the measurements shown below in the *center* of a sheet of paper.

Extend the lines of the sides CA and CB, marking two points D and E on them as shown below.

Measure each of the following angles:

20. ∠DAB.

21. ∠ABE.

22. ∠DCE.

Now use your straightedge and compass to bisect ∠DAB and ∠ABE. Draw the two bisectors long enough so that they meet in a point; label it F.
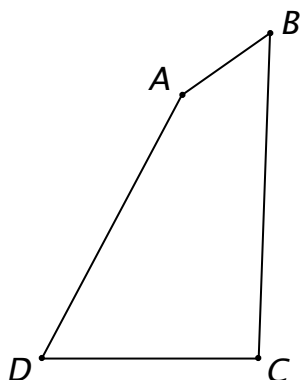
Check the accuracy of your drawing by measuring these angles:

23. ∠DAF and ∠FAB.

24. ∠ABF and ∠FBE.

Draw CF.

25. Measure ∠DCF and ∠FCE.

26. What does the line CF seem to do to ∠DCE?

Summary and Review **37**

27. Draw a quadrilateral of about this shape on your paper.

Use your straightedge and compass to bisect the four sides of the quadrilateral. Use the letter M to label the midpoint of side AB, the letter N for the midpoint of BC, the letter O for the midpoint of CD, and the letter P for the midpoint of DA.

Connect M, N, O, and P with line segments to form quadrilateral MNOP.

28. What do you notice about the sides of this quadrilateral?

29. What is its perimeter in centimeters?

Draw AC and BD. These line segments are called the *diagonals* of quadrilateral ABCD.

30. Measure AC and BD in centimeters and add their lengths.

If you have drawn and measured everything accurately, your answers to exercises 29 and 30 should be the same, even though your original copy of quadrilateral ABCD may not look exactly like the one above!
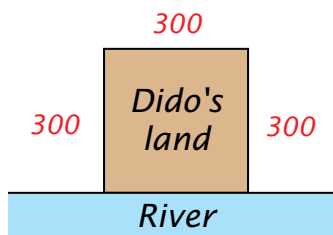
Set III

According to the Roman poet Virgil, a princess named Dido wanted to buy some land in Africa in about 900 B.C. She was told that she could buy as much land as she could enclose with the skin of an ox.

Dido is said to have had the hide cut into thin strips which were then tied together to form as long a cord as possible.

Suppose that her plot of land had to be in the shape of a rectangle, that it could face a river so that the cord would have to stretch only along three sides, and that the cord was 900 yards long.

If the land were in the shape of a square, it would look like this.

1. Including the fourth side, what is the perimeter of this property?

2. What is its area?

The following figure shows another possible shape for Dido's land.

3. In this case, how far would the cord reach across the top side?

4. What is the perimeter of this property?

5. What is its area?

6. What length and width should the rectangle have for Dido to get *as much* land as possible? Make some more drawings to try to find out.

Simplifying Expressions

The operations of addition and multiplication have these basic properties:

	Addition	*Multiplication*
Commutative	$a + b = b + a$	$ab = ba$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	$a + 0 = a$	$1 \cdot a = a$
Inverse	$a + (-a) = 0$	$a \cdot \dfrac{1}{a} = 1$

The *commutative property* tells you that the result of operating on two numbers is independent of their order.

The *associative property* tells you that the result of operating on three numbers is independent of the way in which they are grouped.

The *additive identity* is 0 and the *multiplicative identity* is 1.

The *additive inverse* of a number, a, is its *opposite*, $-a$, and the

multicative inverse of a number, a, is its *reciprocal,* $\dfrac{1}{a}$.

Multiplication can often be understood as *repeated addition*; for example, $3a$ means $a + a + a$. The 3 in the expression $3a$ is called the *coefficient*.

Raising to a power can often be understood as *repeated multiplication;* for example, a^3 means $a \cdot a \cdot a$. The 3 in the expression a^3 is called an *exponent*.

Subtraction and Division
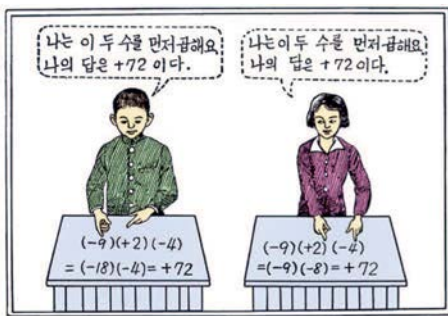
The operations of subtraction and division are not listed above because they are defined in terms of addition and multiplication, respectively.

Subtracting a number is defined as *adding its opposite*.

Example: $a - b = a + (-b)$

Dividing by a number is defined as *multiplying by its reciprocal*.

Example: $\dfrac{a}{b} = a \cdot \dfrac{1}{b}$

나는 이 두 수를 먼저 곱해요. 나의 답은 +72 이다.

나는 이 두 수를 먼저 곱해요. 나의 답은 +72 이다.

$(-9)(+2)(-4)$
$= (-18)(-4) = +72$

$(-9)(+2)(-4)$
$= (-9)(-8) = +72$

This figure illustrating the associative property of multiplication is from a Korean mathematics book.

Exercises

Name the property or definition illustrated by each of the following equations.

1. $(x + 1) + 2 = x + (1 + 2)$
2. $3 \cdot 4 = 4 \cdot 3$
3. $x - (-5) = x + 5$
4. $100 + 0 = 100$
5. $\dfrac{x}{6} = x \cdot \dfrac{1}{6}$
6. $7 \cdot \dfrac{1}{7} = 1$

Write each of the following expressions as a single integer.

7. 12^3
8. 12×10^3
9. $5 - (-15)$
10. $5(-15)$
11. $17^2 - 8^2$
12. $(17 - 8)^2$

Simplify the following expressions.

13. $\pi(7)^2$
14. $2\pi(7)$
15. $x + x + x + x + x$
16. $x \cdot x \cdot x \cdot x \cdot x$
17. $2x + 3x$
18. $x^2 \cdot x^3$
19. $x + x + y + y + y$
20. $x \cdot x \cdot y \cdot y \cdot y$
21. $x \cdot x + y \cdot y \cdot y$
22. $x \cdot x \cdot x + y \cdot y$
23. $(x + y) + y$
24. $(xy)y$
25. $(x^2 + x^2 + x^2 + x^2) + (x^2 + x^2 + x^2)$
26. $4x^2 + 3x^2$
27. $(x^2 + x^2 + x^2 + x^2 + x^2 + x^2) - x^2$
28. $6x^2 - x^2$
29. $(x^2 + x^2 + x^2 + x^2) + (x + x + x)$
30. $x^3 + x^3 + x^3 + x^2 + x^2 + x$
31. $(x^2 \cdot x^2 \cdot x^2 \cdot x^2)(x^2 \cdot x^2 \cdot x^2)$
32. $(x^8)(x^6)$
33. $(2x^3)(3x^2)$
34. $(x^3 + x^3)(x^2 + x^2 + x^2)$
35. $(2x + 2x + 2x)(3x + 3x)$
36. $(2x \cdot 2x \cdot 2x)(3x \cdot 3x)$
37. $(x + x + x + y + y) + (x + y + y)$
38. $(3x + 2y) + (x + 2y)$
39. $(x + x + x + y + y) - (x + y + y)$
40. $(3x + 2y) - (x + 2y)$
41. $(x + y) + -(x + y)$
42. $(x + y)\dfrac{1}{x + y}$
43. $(4x + 7y) + (x + 3y)$
44. $(4x + 7y) - (x + 3y)$
45. $(4x - 7y) + (x - 3y)$
46. $(4x - 7y) - (x - 3y)$
47. $(4x)(7y) + (x)(3y)$
48. $(4x)(7y) - (x)(3y)$
49. $(2 + x^3) + (5 + x^5)$
50. $(2x^3)(5x^5)$

Chapter 8

Transformations

How does this picture illustrate both laughing and crying?

Mathematics has been described as the "science of patterns." Geometry concerns patterns of shape and form. You are already familiar with the pattern of form called line symmetry. In this chapter, we will explore other types of symmetry and become acquainted with their relation to the subject of transformations, including reflections, translations, rotations, and glide reflections. We will also apply these ideas to widen our notion of congruence.

A reflection

8. The largest of
the Rocs picks her
up by the skirt.

5.

Just as he reaches a small grassy point of land, another fish attacks him, lashing furiously with his tail.
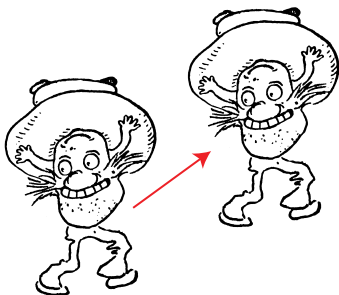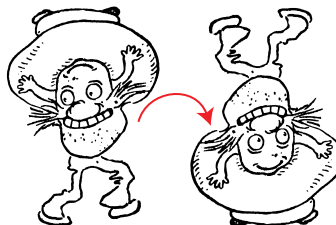
Transformations

Perhaps the most remarkable comic strip ever drawn appeared in the *New York Herald* from 1903 to 1905. A panel from one of the strips is shown above.

The other figures on this page show how several transformations important in geometry would affect one of the strip's characters, Old Man Muffaroo. The rotation is remarkable in that it turns Muffaroo into the strip's other character, Little Lady Lovekins. After the six panels of the comic strip were read in the usual way, the strip had to be turned upside down to read the rest of the story.
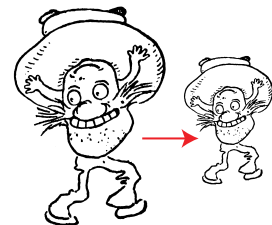
The word "transformation" has a special meaning in geometry. It refers to a rule for getting one set of points from another. When the rule is applied to a point in the first set, it produces exactly one point in the second set. Going backward, we find that, for each point in the second set, there corresponds exactly one point in the first set.

A translation

A rotation

A dilation
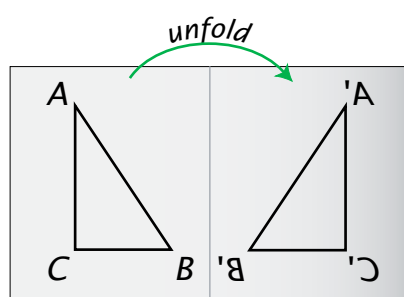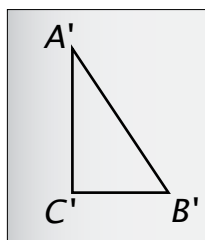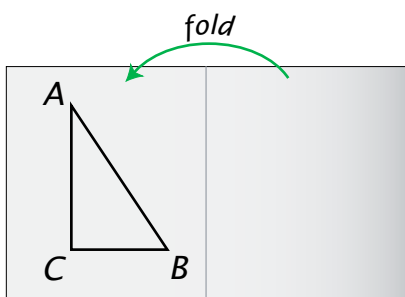
Definition

A *transformation* is a one-to-one correspondence between two sets of points.

Transformations can be applied to triangles. First, imagine tracing △ABC on a sheet of transparent paper and then sliding the paper a certain distance in a given direction without turning it, as shown at the right.

The result is a *translation* of △ABC. Each point in the new figure, △A′B′C′, corresponds to a point in the original figure, △ABC, and is called its *image*. Point A′ (read as "A prime"), for example, is the image of point A.
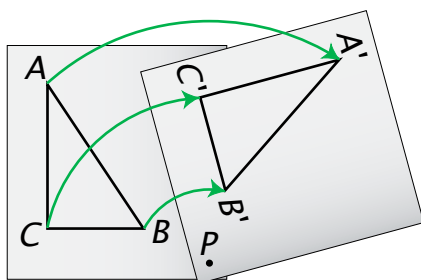
Next, imagine taking the sheet of paper with the drawing of △ABC, folding the paper, tracing △ABC to produce △A′B′C′, and then unfolding the paper, as shown in the three figures below.

A translation
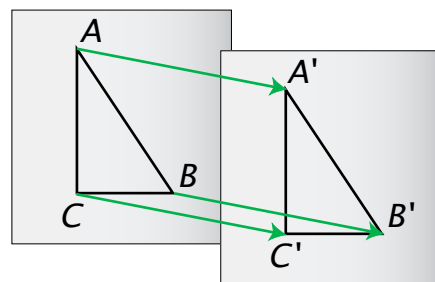
A reflection

The result is a *reflection* of △ABC, and the line along which the paper is folded is called the *mirror* of the reflection.

Now imagine taking the paper with the tracing of △ABC and rotating it a certain number of degrees about a fixed point, P, as shown below.
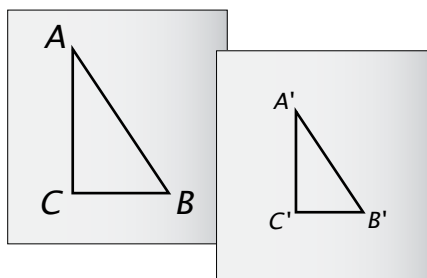
A rotation

This time the result is a *rotation* of △ABC. The fixed point, P, about which the figure is rotated is called the *center* of the rotation.

Finally, imagine taking the paper with the drawing of △ABC and making a photocopy of it by either enlarging it or reducing it, as shown below.

A dilation

The result is a ***dilation*** of △ABC. An obvious but significant difference between this transformation and the preceding ones is that △ABC and its image, △A′B′C′, are not congruent. Although the corresponding angles of the two triangles seem to be equal, the corresponding sides clearly are not. Transformations in which neither distances nor angle measures change are called *isometries*.

Definition
An ***isometry*** is a transformation that preserves distance and angle measure.

From our descriptions, it is evident that the transformations called *translations, reflections,* and *rotations* are examples of isometries. Dilations are not.

Exercises

Set I

Transformations in Art. The figure by Maurits Escher at the right consists of fish of four colors swimming in four directions.

What type of transformation seems to relate

1. two fish of the same color?
2. a pair of red and white fish?
3. a pair of blue and white fish?
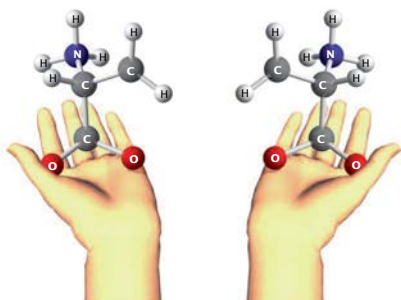
Are there any pairs of fish in the figure for which one fish of the pair seems to be

4. a dilation of the other?
5. a reflection of the other?

Mirror Molecules. The two hands in the figure below are shown holding models of a pair of amino acid molecules. Of the two molecules, L-alanine and D-alanine, only the one on the left is present in natural protein.*

6. Under what transformation is the one hand the image of the other?

7. L and D stand for "levo-" and "dextro-." What do you think these prefixes mean?

Reflections. Children sometimes write the mirror image Я when they mean to write an R. This mistake has been used in the trademark of a toy store.
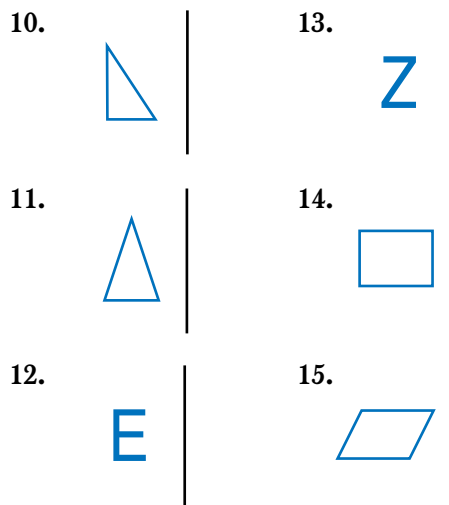
The figure below shows an R and its reflection through a mirror line.

On your paper, copy and complete each of the following figures by including the reflection image of the object on the left through the mirror line.

8.

N |

9.

A |

_____

The Galactic Club, by Ronald N. Bracewell (W. H. Freeman and Company, 1975).

10.

11.

12.

E |

13.

Z |

14.

15.

16. In which of exercises 8 through 15 do the figure and its mirror image look the same?

17. What is it about these figures that causes them and their mirror images to look the same?
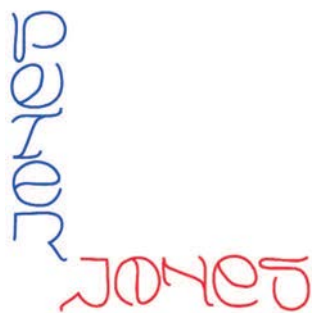
Down the Stairs. The figure below illustrates a transformation.†

18. What transformation is it?

19. What does the word "transformation" mean in geometry?

20. Is this transformation an example of an isometry?

21. What does the word "isometry" mean?

_____

†*Mind Sights,* by Roger N. Shepard (W. H. Freeman and Company, 1990).
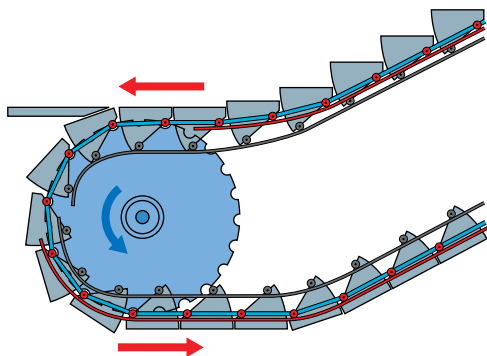
Peter Jones. In this remarkable design by Douglas Hofstadter, the part shown in red is the image of the part shown in blue.

22. What transformation does the figure illustrate?
23. Describe how the image is produced from the original figure.
24. What letter is the image of P?

Set II

Escalator Transformations. The diagram below illustrates how an escalator works.*
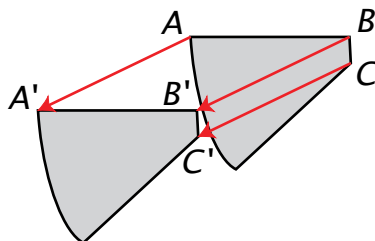
What type of transformation is illustrated by the steps as

25. they descend?
26. they move around the return wheel?

—————————
Scientific American: How Things Work Today, edited by Michael Wright and M. N. Patel (Crown, 2000).

The figure below represents a translation of one step of an escalator; AA′ ∥ BB′ ∥ CC′ and AA′ = BB′ = CC′.
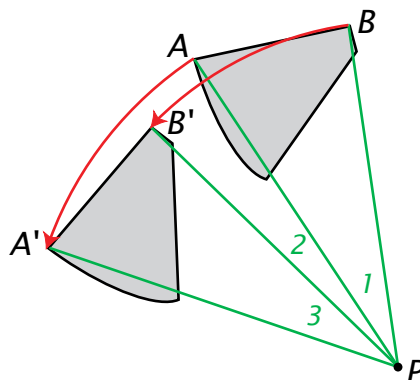
Let's see how this transformation affects distances and angles.

27. Copy the figure and mark it as needed to do each of the following exercises.
28. What can you conclude about quadrilaterals AA′B′B and BB′C′C? Why?
29. Why is A′B′ ∥ AB?
27. *(continued)* Draw AC and A′C′.
30. What can you conclude about quadrilateral AA′C′C?
31. Why is AB = A′B′, BC = B′C′, and AC = A′C′?
32. Why is △ABC ≅ △A′B′C′?
33. Why is ∠ABC = ∠A′B′C′?

The answers to exercises 31 and 33 establish that the translation part of the escalator motion is an isometry.

34. What is an isometry?
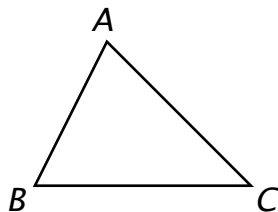
The figure below represents the rotation of one step; PA = PA′, PB = PB′, and ∠APA′ = ∠BPB′.

Let's see how this transformation affects distances.

35. Copy the figure and mark it as needed to answer each of the following questions.

36. Why is $\angle APA' = \angle 2 + \angle 3$ and $\angle BPB' = \angle 1 + \angle 2$?

37. Why is $\angle 2 + \angle 3 = \angle 1 + \angle 2$?

38. Why is $\angle 3 = \angle 1$?

39. Why is $\triangle ABP \cong \triangle A'B'P$?

40. Why is $AB = A'B'$?

41. What have we shown to be preserved by this rotation?

Triangle Construction. The figure below can be used to construct a transformation of $\triangle ABC$.
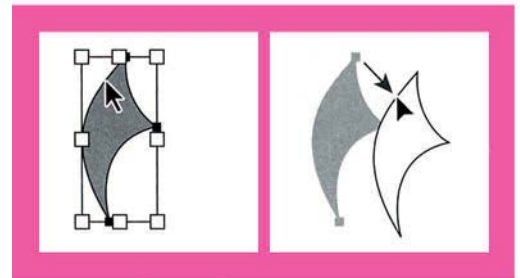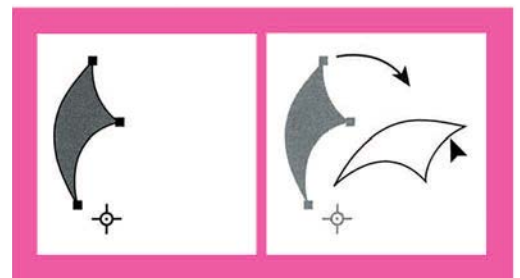
42. Make a large copy of the figure. Draw ray PA. Use your compass to mark a point A′ on PA such that $AA' = PA$. Draw ray PB and find point B′ on it so that $BB' = PB$. Draw ray PC and find point C′ on it so that $CC' = PC$. Draw $\triangle A'B'C'$.

43. What type of transformation does your drawing seem to illustrate?

44. How do the sides of $\triangle A'B'C'$ seem to compare in length with the sides of $\triangle ABC$?

45. How do the angles of $\triangle A'B'C'$ seem to compare in size with the angles of $\triangle ABC$?

46. Is this transformation an isometry? Explain.

Computer Geometry. Transformations are used in computer drawing programs. The following figures appear in the Adobe Illustrator User Guide.* Name the transformation illustrated in each figure.
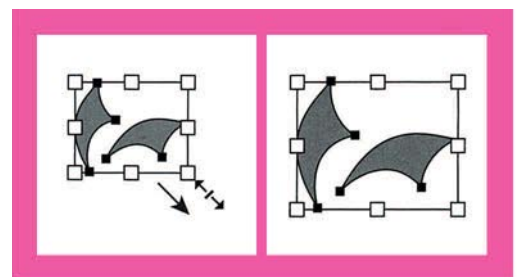
47.

48.

49.

(In this figure, the drawing on the right is the transformation image of the drawing on the left.)

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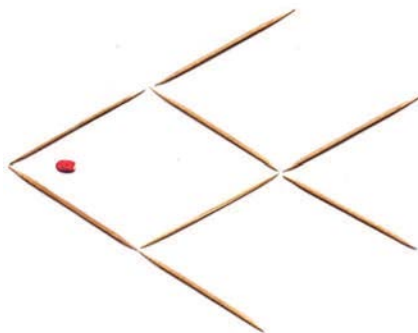
*Adobe Illustrator 9.0 User Guide (Adobe Systems, 2000).

Computer drawing programs transform objects by changing their coordinates. To show, for example, that the coordinates of each point of an object are to be reversed to get the coordinates of each point of its image, we can write $(a, b) \rightarrow (b, a)$.

50. On graph paper, draw a pair of axes extending 10 units in each direction from the origin. Plot the following points and connect them to form $\triangle ABC$: A(3, 1), B(5, 2), C(2, 6).

51. Use the transformation $(a, b) \rightarrow (a + 2, b - 7)$ on the coordinates of $\triangle ABC$ to find the coordinates of $\triangle DEF$. For example, $A(3, 1) \rightarrow D(3 + 2, 1 - 7)$, or D(5, −6).

50. *(continued)* Draw $\triangle DEF$.

52. For what type of transformation is $\triangle DEF$ the image of $\triangle ABC$?

53. Use the transformation $(a, b) \rightarrow (-a, b)$ on the coordinates of $\triangle ABC$ to find the coordinates of $\triangle GHI$. For example, $A(3, 1) \rightarrow G(-3, 1)$.

50. *(continued)* Draw $\triangle GHI$.

54. For what type of transformation is $\triangle GHI$ the image of $\triangle ABC$?

55. Use the transformation $(a, b) \rightarrow (-a, -b)$ on the coordinates of $\triangle ABC$ to find the coordinates of $\triangle JKL$.

50. *(continued)* Draw $\triangle JKL$.

56. For what type of transformation is $\triangle JKL$ the image of $\triangle ABC$?

57. Use the transformation $(a, b) \rightarrow (2a, 2b)$ on the coordinates of $\triangle ABC$ to find the coordinates of $\triangle MNO$.

50. *(continued)* Draw $\triangle MNO$.

58. For what type of transformation is $\triangle MNO$ the image of $\triangle ABC$?

Set III

Toothpick Puzzle. A puzzle thought to have originated in Japan uses eight toothpicks and a matchhead.*

The challenge is to move just three toothpicks and the matchhead to make the fish swim in the opposite direction.

1. Can you do it? If so, make a drawing illustrating the original fish and what you did to get your answer.

2. Can the solution be regarded as the result of one of the transformations considered in this lesson? If so, which one? Explain.
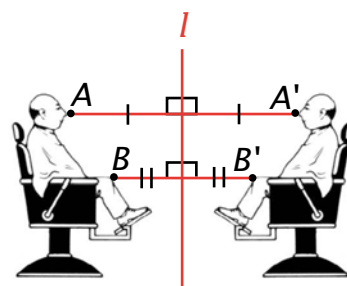
Reflections

Some barber shops have mirrors on the walls that face each other, giving a series of reflections as seen above. After dozing during his haircut, the man in the barber chair still sees a remnant of his bad dream.

You already know that a reflection is a special type of transformation. The figure at the right shows how to find the points corresponding to A and B when an object is reflected in a mirror, line l. Find the perpendicular to l through the point, go to the mirror, and continue an equal distance on the other side. The distance from any point *on l to l* is 0; so the reflection of such a point through the line is the point itself.

A reflection
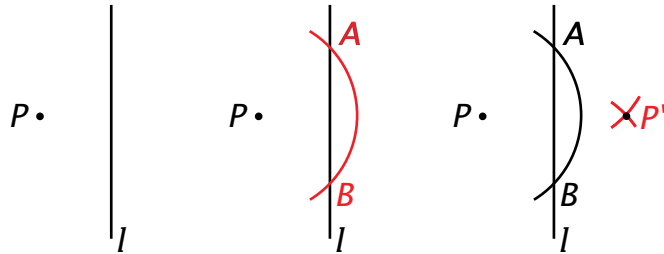
Definition

The ***reflection*** of point P through line l is P itself if P lies on l. Otherwise, it is the point P′ such that l is the perpendicular bisector of PP′.
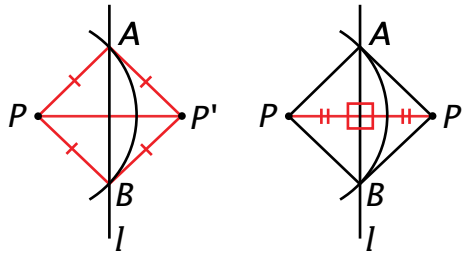
The reflection of a point through a line can be found by using a mirror or by folding. It can also be found by construction.

Construction 8
To reflect a point through a line.
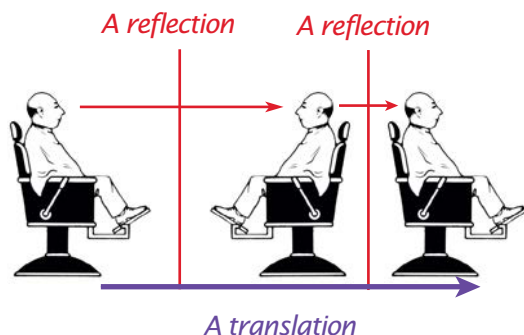
Let the point be P and the line be l. With P as center, draw an arc that intersects l in two points, A and B. With A and B as centers, draw two more arcs with the same radius as the first arc. The point in which they intersect, P′, is the reflection of P through l.

These figures show why this method works. Because PA = PB = AP′ = BP′, points A and B are equidistant from P and P′. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment; so l is the perpendicular bisector of PP′. It follows from the definition of the reflection of a point through a line that P′ is the reflection of P through l.

In the cartoon by Charles Addams at the beginning of this lesson, not all of the successive images produced by the parallel mirrors are alike. Some of the images of the man in the barber chair face in the same direction and some face in the opposite direction. The second image, which faces in the same direction as the man in the chair, is the result of another special transformation, a *translation*. It's as if the man in the chair could be pushed through two "walls" to coincide with his translation image. We have already described a translation as sliding without turning. The figure suggests that a translation can be produced by two successive reflections in parallel mirrors.

A reflection *A reflection*

A translation

We will call a transformation that is the result of two or more successive transformations their *composite* and define translation in the following way.

Definition

A ***translation*** is the composite of two successive reflections through parallel lines.

The distance between a point of the original figure and its translation image is called the *magnitude* of the translation. The length of the blue arrow in the figure above illustrates the magnitude of the translation.
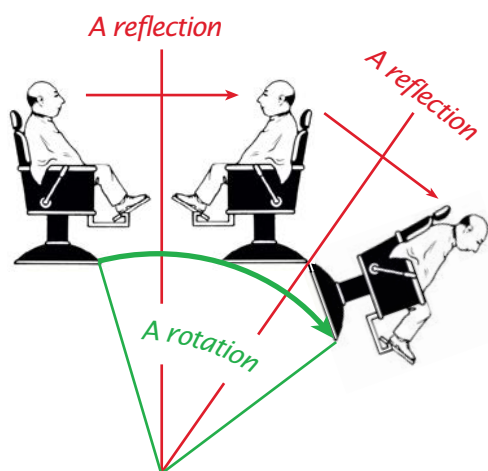
A translation is the composite of two reflections through *parallel* lines. What would the composite of two reflections through *intersecting* lines be?

The figure at the right suggests that it would be a rotation. If the mirrors in the barbershop were not parallel, the customer might get the impression that the barber was trying to tip him out of the chair!

Definition

A ***rotation*** is the composite of two successive reflections through intersecting lines.

A reflection *A reflection*

A rotation

The point in which the lines intersect is the *center* of the rotation, and the measure of the angle through which a point of the original figure turns to coincide with its rotation image is called the *magnitude* of the rotation. The green angle in this figure illustrates the magnitude of the rotation.

Exercises

Set I

A Suspicious Cow. Take a good look at this photograph of a cow and barn.

1. What is the "mirror" that causes the reflection?

2. What is misleading about the photograph?

3. Make a large copy of the figure below. Use your straightedge and compass to construct the reflections of points A and B through line *l*.
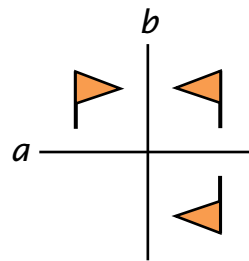
4. If a point is *x* units from a mirror in which it is reflected, how far is it from its image?

5. What relation does the mirror have to the line segment that connects a point and its image?
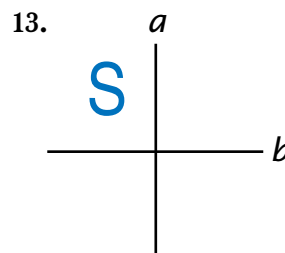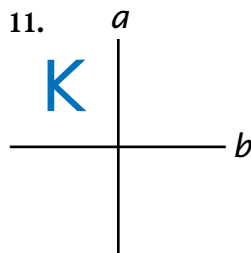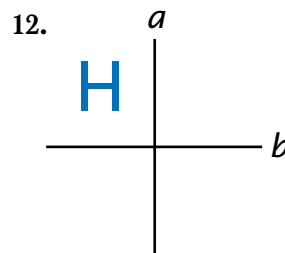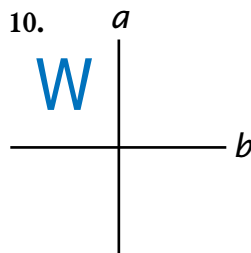
Double Reflections. It is fairly easy to imagine a reflection through a vertical or horizontal line because mirrors are usually vertical or horizontal.

The figure at the right shows two successive reflections of the flag at the upper left.

6. Through which line was the flag reflected first?

7. Through what transformation is the flag at the lower right the image of the original flag?

8. What does the measure of the angles formed by the two lines appear to be?

9. What does the magnitude of the transformation from the first flag to the last flag appear to be?

Copy the following figures on your paper. Sketch the reflection of each letter through line *a* (a vertical line) and then sketch the reflection of the image that results through line *b* (a horizontal line).

10.

11.

12.

13.
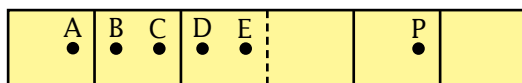
Look at your drawings for exercises 10 through 13 to answer the following questions.

What happens when a figure with

14. a vertical line of symmetry is reflected through a vertical line?

15. a horizontal line of symmetry is reflected through a horizontal line?

16. point symmetry is reflected through both a vertical and a horizontal line?

17. In which exercise do all three figures look alike? Why?
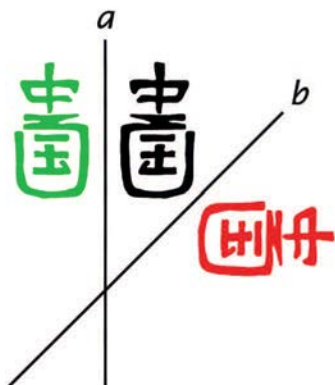
SAT Problem. The following figure appeared in a problem on an SAT exam.

All of the boxes in the strip are the same size.

18. When the strip is folded together along the dashed line, which point is most likely to coincide with point P?

19. What transformation does this problem illustrate?

20. How could you use a ruler to check your answer?

Can You Read Chinese? The figure in green below was designed by David Moser when he was a graduate student in Chinese.*

Through what transformation is

21. the figure in black the image of the figure in green?

22. the figure in red the image of the figure in black?

23. the figure in red the image of the figure in green?

———————

Can You Believe Your Eyes? by J. R. Block and Harold E. Yuker (Brunner/Mazel, 1992).
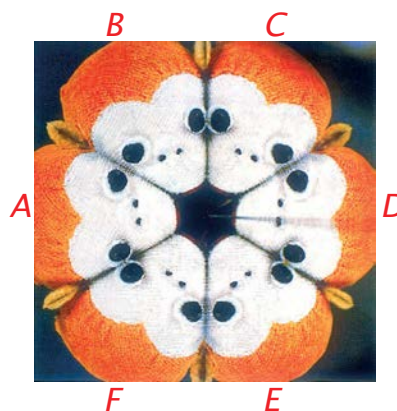
A reader of Chinese would read the figure in green as the word "China."

24. Would it be correct to say that the figure in red is a rotation of the figure in green? Explain.

25. In what sense would it be correct to say that the figure in red is a translation of the figure in green?

Set II
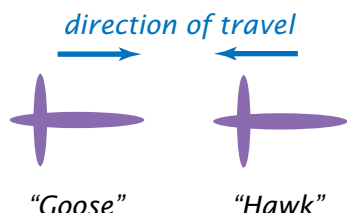
Kaleidoscope Patterns. A kaleidoscope uses mirrors to produce symmetric patterns. The photograph below shows the face of a toy monkey and five of its images in a kaleidoscope with two mirrors that meet at a 60° angle.
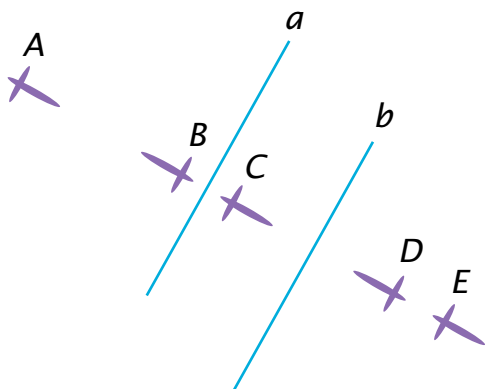
The monkey's face is at A. Which of the five images are

26. reflections of it?

27. rotations of it?

28. What is the magnitude of each rotation?

29. How many lines of symmetry does the pattern of monkey faces seem to have? Where are they?

30. Does the pattern of faces seem to have point symmetry? Explain why or why not.

Scaring Chickens. Experimental findings have shown that, if chickens see this shape flying overhead in the direction of its long end, they ignore it. If they see it flying in the direction of its short end, they run for cover.*

direction of travel

"Goose"   "Hawk"

In the figure below, $a \parallel b$ and birds A, B, D, and E are reflection images of bird C through either or both of the lines.

Which bird is the reflection image of

31. bird C through *a?*

32. bird B through *b?*

33. bird C through *b?*

34. bird D through *a?*
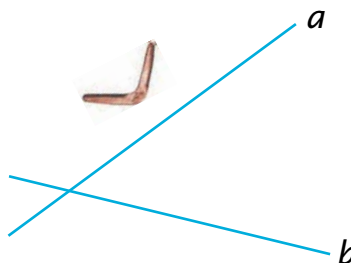
Which bird is the image of bird C as a result of successive reflections through

35. *a* and *b?*

36. *b* and *a?*

37. What transformation do exercises 35 and 36 illustrate?

_____
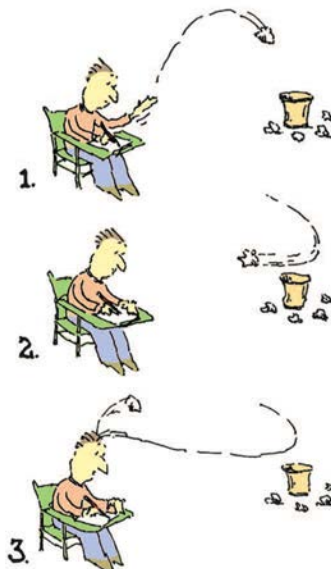
Visual Intelligence: How We Create What We See, by Donald D. Hoffman (Norton, 1998).

38. Use the definition of that transformation to explain why.

39. Which would scare chickens: seeing the bird at C flying to A or seeing it fly to E?

Boomerang. The figure below shows a boomerang flying overhead.
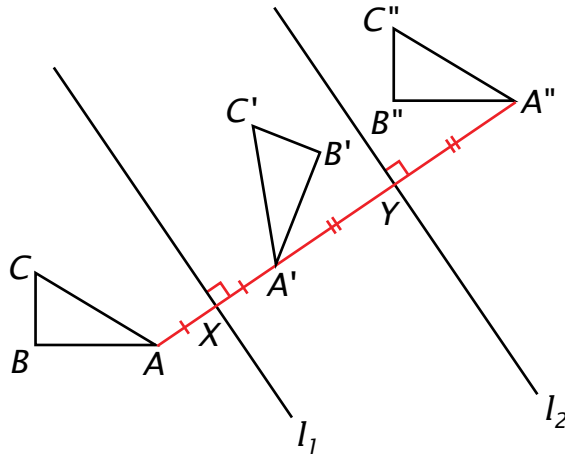
40. Draw a large copy of the figure and sketch the reflection of the boomerang through line *a*. Then sketch the reflection of the resulting image through line *b*. (Doing so will be easier if you turn your paper so that each line is vertical as you reflect through it.)

41. Under what transformation is the final image the image of the original figure?

42. Use the definition of that transformation to explain why.

43. Where is the center of the transformation?

4.

Triangle Reflections. In the figure below, $\triangle A'B'C'$ is the reflection of $\triangle ABC$ through l_1 and $\triangle A''B''C''$ is the reflection of $\triangle A'B'C'$ through l_2; $l_1 \parallel l_2$.

44. Why is $AX = XA'$ and $A'Y = YA''$?

45. Through what transformation is $\triangle A''B''C''$ the image of $\triangle ABC$?

46. What is the length of AA'' called with respect to this transformation?

47. How does the length of AA'' compare with that of XY, the distance between l_1 and l_2?

In the figure at the right, $\triangle A'B'C'$ is the reflection of $\triangle ABC$ through l_1 and $\triangle A''B''C''$ is the reflection of $\triangle A'B'C'$ through l_2; l_1 and l_2 intersect at point O.

48. Why is $\triangle AOX \cong \triangle A'OX$ and $\triangle A'OY \cong \triangle A''OY$?

49. Why is $OA = OA'$ and $OA' = OA''$?

50. Through what transformation is $\triangle A''B''C''$ the image of $\triangle ABC$?

51. What is the measure of $\angle AOA''$ called with respect to this transformation?

52. How does the measure of $\angle AOA''$ compare with that of $\angle XOY$, the angle between l_1 and l_2?

Set III

What Time Was It? In the murder mystery titled *The House of the Arrow* by A. E. W. Mason, the solution depends on a clock:

The three witnesses advanced into the room, and as they looked again, from close at hand and with a longer gaze, a cry of surprise broke from all of them.
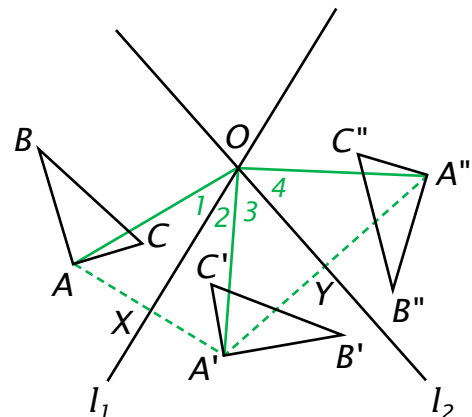
There was no clock upon the marquetry cabinet at all.
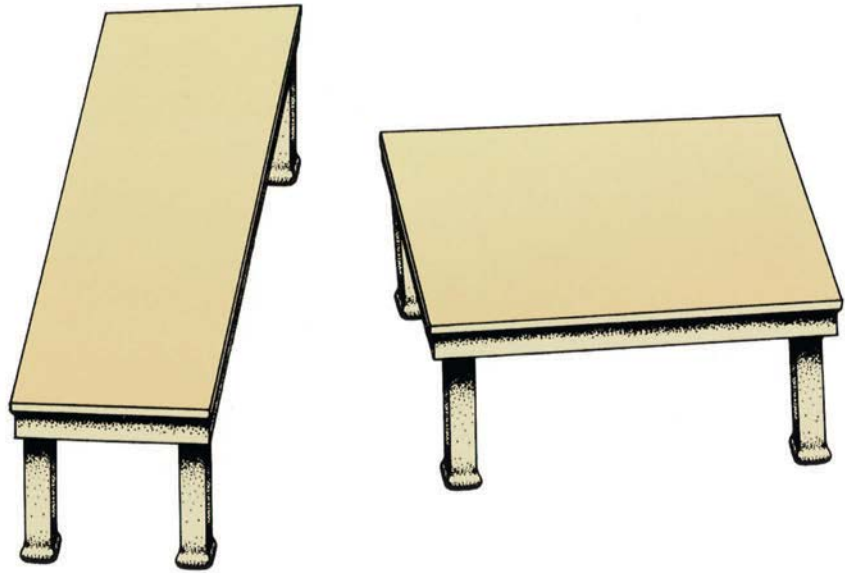
But high above it in the long mirror before which it stood there was a reflection of a clock, its white face so clear and bright that even now it was difficult to disbelieve that this was the clock itself. And the position of the hands gave the hour as precisely half-past ten.

"Now turn about and see!" said Hanaud.

The clock itself stood upon the shelf of the Adam mantelpiece and there staring at them, the true hour was marked. It was exactly. . . .

1. Exactly what time was it? Make some drawings to support your answer.

2. Are there any other times when the reflection would also look like an actual time? Explain.

Isometries and Congruence

Here is an amazing illusion by psychologist Roger Shepard. How does the shape of the table on the left above compare with the shape of the table on the right? If you trace the outline of the top of one of the tables on a thin sheet of paper and place it on the other table, you will find that it fits exactly. In other words, the tabletops are congruent!*

When this method is used to test for congruence, the tracing of the tabletop is *translated* and *rotated* to see if it can be made to coincide with the other tabletop. Because translations and rotations are isometries, this procedure suggests that congruence can be defined as an isometry. This definition agrees with our earlier definition of congruence for triangles, and it extends the idea so that it can be applied to any pair of figures having the same size and shape.

Definition
Two figures are *congruent* if there is an isometry such that one figure is the image of the other.

Mind Sights, by Roger N. Shepard (W. H. Freeman and Company, 1990).

Look at the figure at the right showing the tabletops at a smaller scale. The definition of congruence in relation to isometry says that ABCD ≅ EFGH if there is an isometry such that EFGH is the image of ABCD. Translations and rotations are composites of reflections. Is it possible to find a composite of *just reflections* in which EFGH is the image of ABCD?

Notice that E appears to be the image of A. Imagine that the figures are printed on a transparent sheet of paper and that we fold the paper so that A fits on E. The second figure shows what happens. Tracing the reflection image of ABCD through the fold line, l_1, produces EB′C′D′.

Next, since F appears to be the image of B, and hence B′, imagine folding the paper so that B′ fits on F. The third figure shows what happens. The reflection image of EB′C′D′ through the fold line, l_2, is EFGH. The composite of the two reflections through l_1 and l_2 is an isometry in which EFGH is the image of ABCD, so ABCD ≅ EFGH.
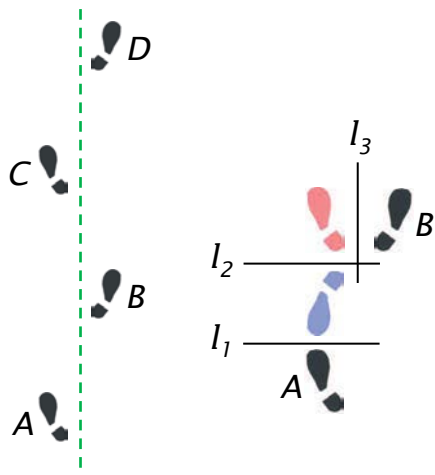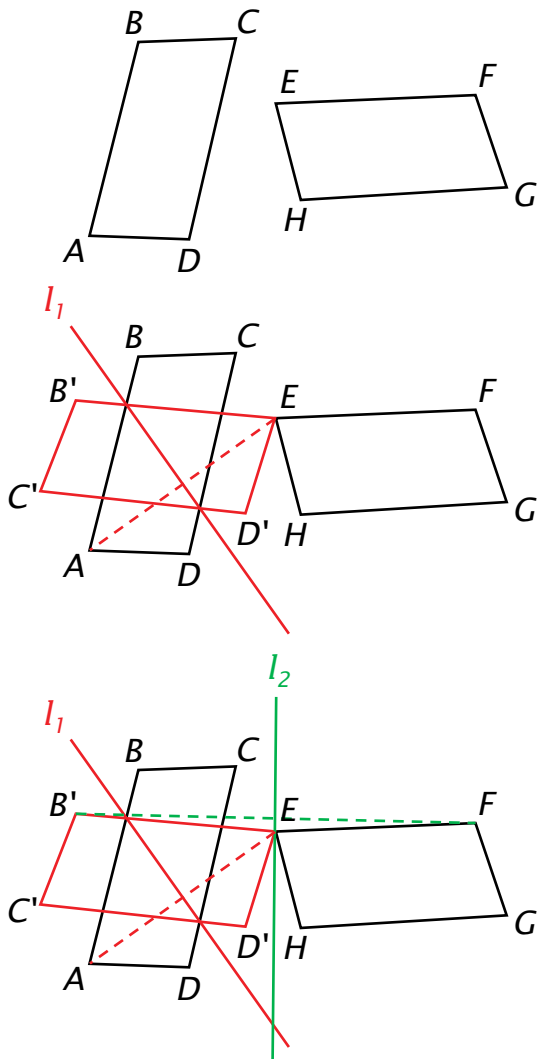
Just two reflections were needed to show that the tabletops are congruent. Is it always possible to get a pair of congruent figures to coincide with two reflections? The figure at the bottom left showing the footprints of successive steps taken by someone walking forward suggests an answer to this question.

Suppose the four steps are congruent. Step C is a translation image of step A, as is step D of step B. Under what transformation is step B an image of step A? It looks like a combination of a translation and a reflection. Such a transformation is called a *glide reflection*.

Definition
A *glide reflection* is the composite of a translation and a reflection in a line parallel to the direction of the translation.

Because a translation is the composite of *two* reflections, a glide reflection is the composite of *three* reflections. Getting step A to coincide with step B is an example of an isometry that requires three reflections. Notice that the left and right steps look different. Each is a *mirror image* of the other because it takes an odd number of reflections to carry either onto the other. We will explore congruence and isometries further in the exercises of this lesson.
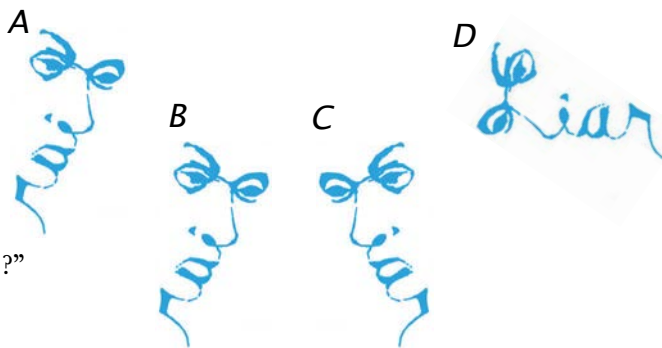
Exercises

Set I

Prevaricator. The four figures at the right by graphic artist Paul Agule are congruent.*

  The artist asks: "Can you trust this man?"
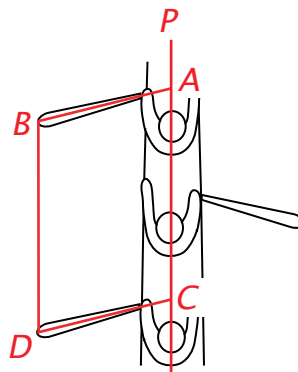
1. What do you think?
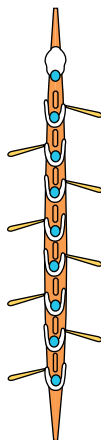
Name the isometry through which

2. figure B appears to be the image of figure A.

3. figure C appears to be the image of figure B.

4. figure D appears to be the image of figure C.

5. figure C appears to be the image of figure A.

6. Is there an isometry through which D is the image of figure B?

7. Use the definition of congruent figures in this lesson to explain why or why not.

Synchronized Oars. For maximum speed, rowers have to keep their oars synchronized. The oars along each side of the boat are synchronized if they are always parallel.

8. Name the two angles in the figure below that *must* be equal if AB ∥ CD.

9. If AB ∥ CD, why is it reasonable to think that ABDC is a parallelogram?

10. Why does it follow that the distance between the tips of the oars, BD, is equal to the distance between the rowers, AC?

What transformation relates the positions of two oars

11. on the same side of the boat?

12. on opposite sides of the boat?

Synchronization is also important in another water sport.
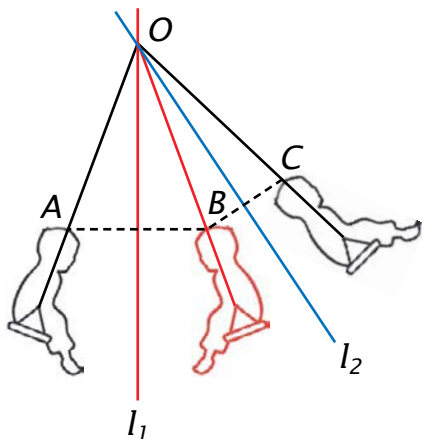
13. What sport is it?

Swing Isometries. Little children like being rotated in a swing.

The figure below illustrates the composite of two isometries.
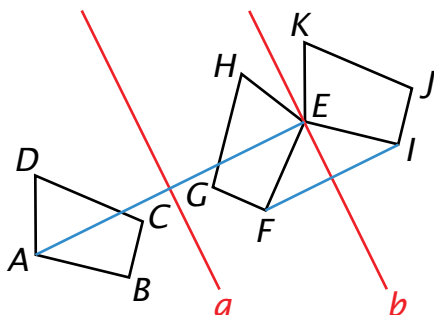
Through what transformation is

14. B the image of A and C the image of B?

15. C the image of A?

16. What is point O called with respect to this transformation?

17. What is the measure of $\angle AOC$ called?

What relation do l_1 and l_2 have to

18. $\angle AOB$ and $\angle BOC$?

19. AB and BC?

20. How does $\angle AOC$ compare in measure with that of the acute angle formed by l_1 and l_2?
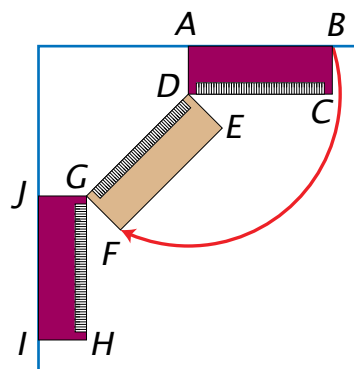
Quadrilateral Reflections. In the figure below, EFGH and EIJK are successive reflection images of ABCD. Lines a and b and line segments AE and FI have been added to the figure to show that ABCD and EIJK are related by a certain isometry.

21. What postulate permits us to draw AE and FI?

22. What relation does line a appear to have to AE and line b to FI?

23. What relation do lines a and b appear to have?

24. If your observations are true, through what transformation is EIJK the image of ABCD?

25. How would this transformation show that ABCD $\cong$ EIJK?

Set II

Moving a Piano. A heavy piano is to be moved from one wall of a room to another. To do so, the piano is turned on its casters, which are at its corners.*

26. What type of transformation is illustrated?

27. What point is its center?

What point on DEFG is the image of

28. point A?

29. point C?

30. point D?

31. What transformation is needed to finish moving the piano? Describe it.

32. Through how many degrees do you think the piano is turned altogether?

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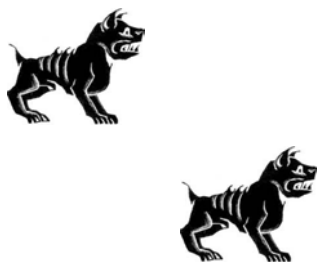
*Adapted from a problem in *The School Mathematics Project, Book 4* (Cambridge University Press, 1969).

Bulldogs. All of the bulldogs in this mosaic by Escher are congruent.

33. What does it mean in relation to isometry to say that two figures are congruent?

34. Through what transformation is one of the bulldogs in the figure below the image of the other?

Is it possible that one of these bulldogs could be the image of the other

35. by a single reflection? Explain.

36. by the composite of two reflections? Explain.

37. by the composite of three reflections? Explain.

38. Through what transformation is one of the bulldogs in the figure at the top of the next column the image of the other?
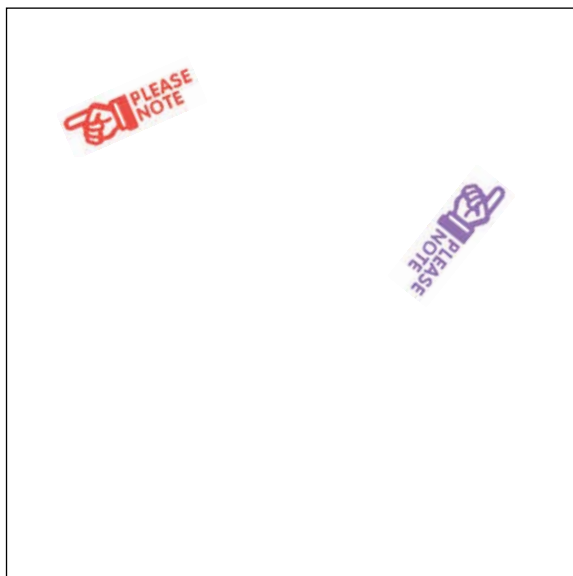
39. Of what *two* transformations is this transformation a composite?

40. Of what *three* transformations is it a composite?

Grid Problem. On graph paper, draw a pair of axes extending 20 units to the right and 15 units up from the origin.

41. Plot the following points and connect them with line segments to form △ABC: A(5, 4), B(8, 3), C(3, 0).

41. *(continued)* Plot point A′(11, 10) and draw AA′. Draw △A′B′C′ so that it is the translation image of △ABC.

42. What are the coordinates of points B′ and C′?

43. Find the magnitude of the translation.

41. *(continued)* Draw the line determined by the points (7, 0) and (20, 13) and label it *l.*

44. What relation does AA′ appear to have to line *l*?

41. *(continued)* Draw △A″B″C″ so that it is the reflection image of △A′B′C′ through line *l.*

45. If your observation in exercise 44 is true, through what transformation is △A″B″C″ the image of △ABC? Explain.

41. *(continued)* Draw AA″, BB″, and CC″ and use the grid to help in finding their midpoints. Label them M, N, and P, respectively.

46. What are the coordinates of M, N, and P?

47. Exactly how is the *y*-coordinate of each of these three points related to the *x*-coordinate?

48. What is interesting about the three points?
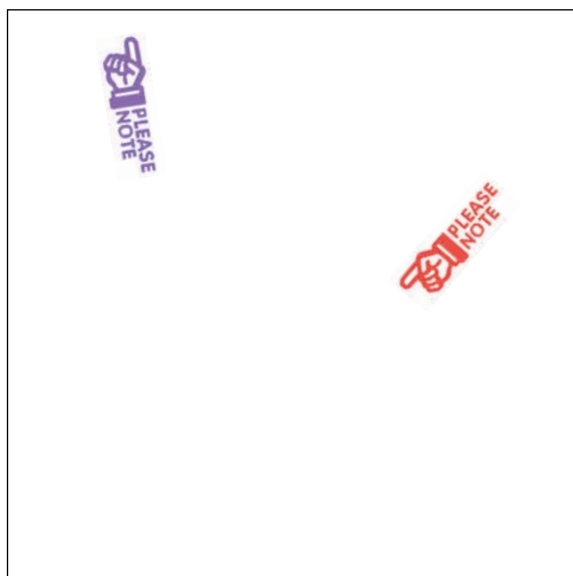
Stamp Tricks. Obtuse Ollie found a rubber stamp and stamped it twice on a sheet of tracing paper as shown in figure 1 below.

1

Acute Alice folded the paper to reflect one of the figures twice, getting the result shown in figure 2 below.

2
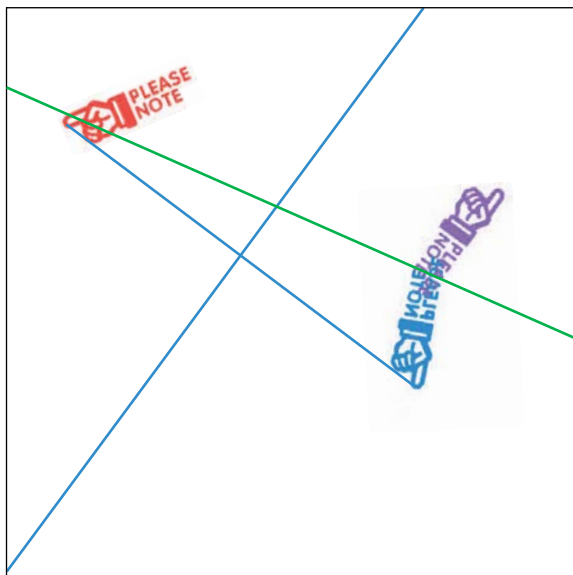
Surprised, Ollie stamped another sheet of tracing paper twice as shown in figure 3 below.

3

Again, Alice managed to fold the paper just twice, getting the result shown in figure 4 below.

4
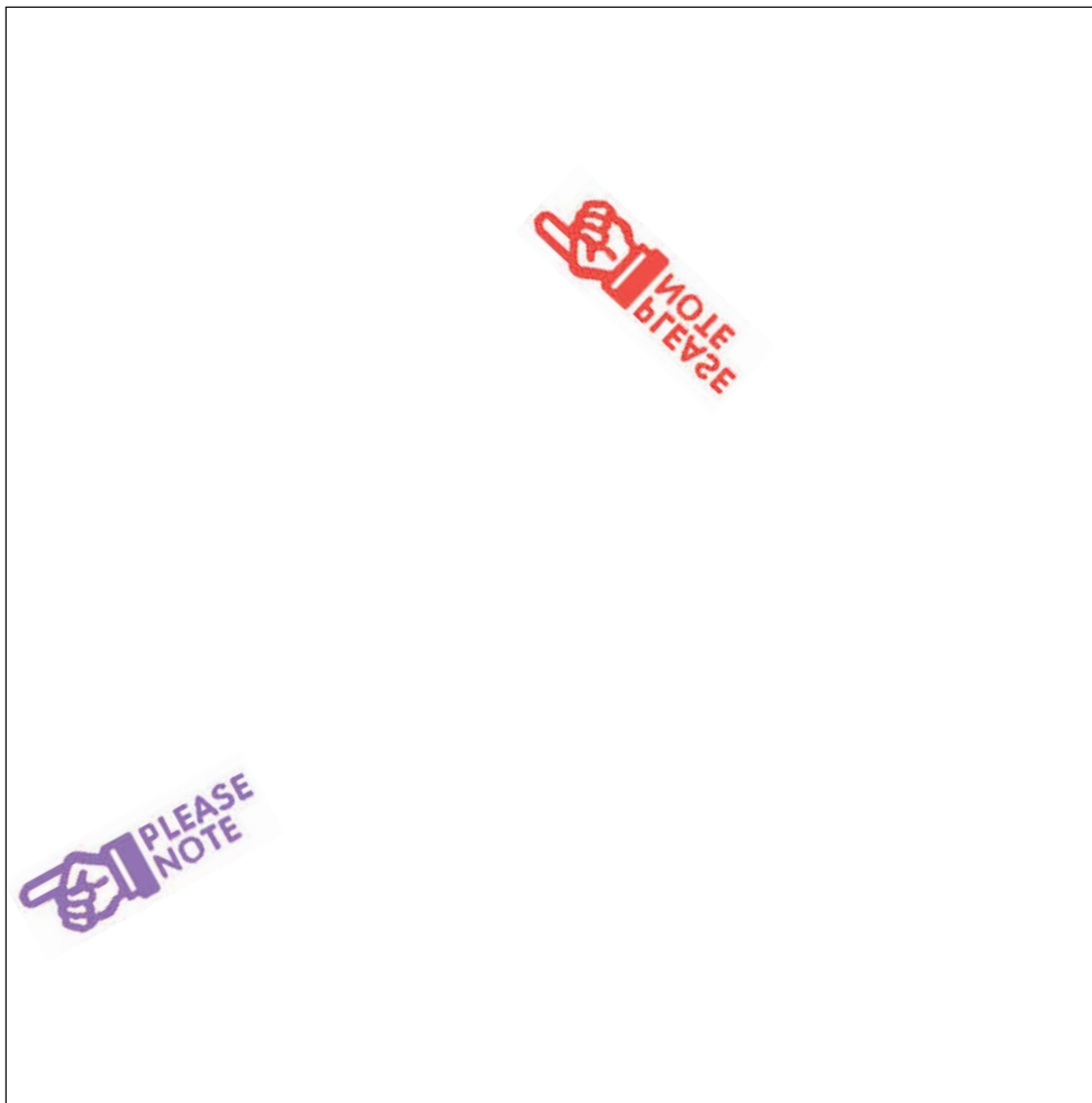
Ollie couldn't believe it. He decided to try to trick Alice and stamped a third sheet of tracing paper as shown below.

This time, Alice had to fold the paper more than two times to get the images to coincide.

1. How did Ollie try to trick Alice?

2. How did his "trick" prevent Alice from being able to get the images to coincide with just two folds?

3. Trace the two figures on a sheet of tracing paper. Can you get them to coincide with three folds?
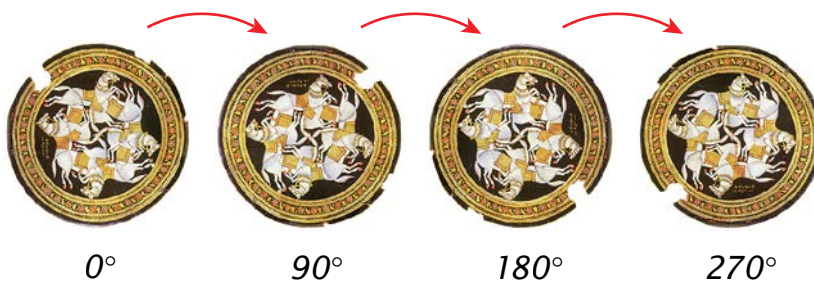
Transformations and Symmetry

How many horses are pictured on the plate shown above? Counting the heads gives a different number from that obtained by counting the bodies.

The creator of the plate, who lived in Persia in the seventeenth century, made use of symmetry in its design. Each time the plate is rotated 90°, as the figures below show, it looks almost the same.

| 0° | 90° | 180° | 270° |
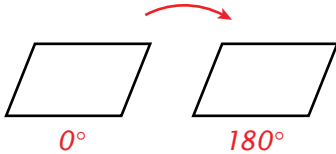
A figure that looks *exactly* the same after being rotated less than one full turn about its center is said to have *rotation symmetry*.

Definition
A figure has *rotation symmetry* with respect to a point iff it coincides with its rotation image through less than 360° about the point.
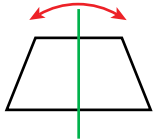
The point, which is the center of the rotation, is called the *center* of the symmetry.

The Persian plate can be rotated so that it looks the same in four positions. For this reason, it has *4-fold rotation symmetry*. Each position corresponds to a turn of 90°: $\frac{360°}{4} = 90°$. In general, a figure has ***n-fold rotation symmetry*** iff the smallest angle through which it can be turned to look exactly the same is $\frac{360°}{n}$.

Recall that a figure that has *point symmetry* looks exactly the same when it is turned upside down. A parallelogram has point symmetry. The figure at the left shows that it also has 2-fold rotation symmetry. So point symmetry is symmetry by a rotation of 180°.

Just as rotation symmetry and hence point symmetry are related to the rotation transformation, *line symmetry* is related to the reflection transformation. For this reason, line symmetry is also called *reflection symmetry*.
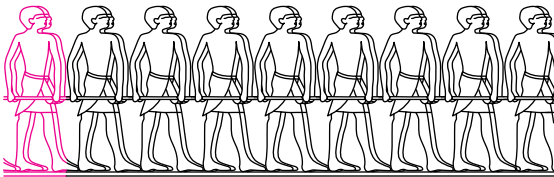
An isosceles trapezoid has reflection symmetry. If the trapezoid is folded along the line shown in the figure at the left, the two halves coincide. Or, if a two-sided mirror is placed on the line, each half of the trapezoid is reflected onto the other half.

We can define reflection (line) symmetry in relation to the reflection transformation.

Definition

A figure has ***reflection (line) symmetry*** with respect to a line iff it coincides with its reflection image through the line.

The line is sometimes called the *axis* of the symmetry.

The translation transformation also is related to symmetry. The figure at the left from ancient Egypt is an example. The part of the drawing shown in red does not have any symmetry of its own but, like the repeating pattern of the horses around the plate, the repeating pattern of the people illustrates a type of symmetry. Rather than repeating around a circle, this pattern repeats along a line. The part of the drawing shown in red is translated a certain distance in the same direction again and again. The resulting pattern has *translation* symmetry. In defining this type of symmetry, we think of the pattern as continuing in both directions endlessly.

Definition

A pattern has ***translation symmetry*** iff it coincides with a translation image.

Because rotations, reflections, and translations are isometries, we know that corresponding distances and angles in symmetric figures are equal.

Exercises
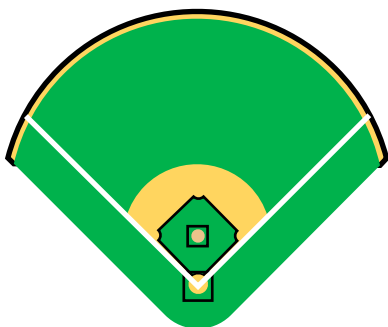
Set I

Ambigrams. The figures below are examples of "ambigrams," words that can be read in more than one way.*
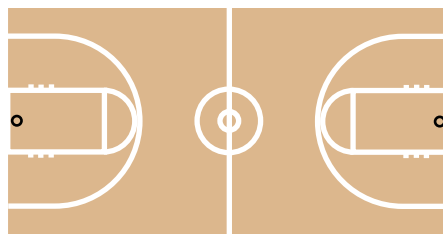
1. What type of symmetry does the first figure have?
2. What is a simple test for this type of symmetry?
3. What type of symmetry does the second figure have?
4. What is a simple test for this type of symmetry?

Sport Symmetry. The fields and courts on which most sports are played are symmetric.

5. What sport is played on the field above?
6. Describe its symmetry.

_____

*"Algebra" is from *Wordplay,* by John Langdon (Harcourt Brace Jovanovich, 1992).

7. What sport is played on the court above?
8. Describe its symmetry.
9. What type of symmetry do you think most sport fields and courts have?
10. Why do you think they have it?

Mount Vernon. George Washington's home, Mount Vernon, has a fake window. It is the second upstairs window from the left.

11. Why do you suppose Washington had this window painted on the outside wall?

Symmetries of Basic Figures. Some geometric figures are so simple that we tend to ignore their symmetry.
   A point has rotation symmetry.
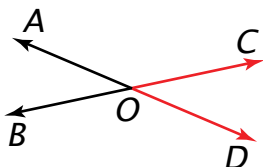
12. Where is its center of symmetry?
13. Does a point have line symmetry? If so, where is the line?

Euclid defined a "straight line" as "a line that lies symmetrically with the points on itself."
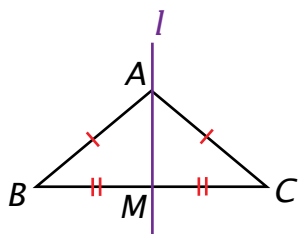
14. A line has rotation symmetry. Explain.
15. What points can serve as its center of symmetry?
16. A line has reflection symmetry. Explain.
17. How many lines of symmetry does a line have? Where are they?
18. Does a line have translation symmetry? Explain.

We defined "vertical angles" with reference to opposite rays.

19. Explain what this definition says about ∠COD and ∠BOA, given that they are vertical angles.
20. ∠COD can be thought of as a rotation image of ∠BOA. Explain.
21. What theorem about vertical angles is suggested by the fact that rotations are isometries?
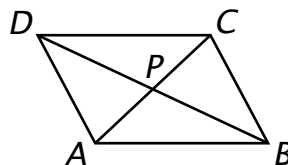
△ABC is an isosceles triangle with AB = AC.

22. If a mirror line l is drawn through A and M, the midpoint of BC, $l \perp$ BC. Why?
23. What are the images of A and B reflected through l?

24. What theorem about isosceles triangles is suggested by the fact that reflections are isometries?

ABCD is a parallelogram with diagonals AC and BD.

25. What must be true about AC and BD?
26. If ABCD is rotated 180° about point P as center, what is the image of AB?
27. What theorem about the sides of a parallelogram is suggested by the fact that rotations are isometries?
28. Does BD appear to be a rotation image of AC? Explain.
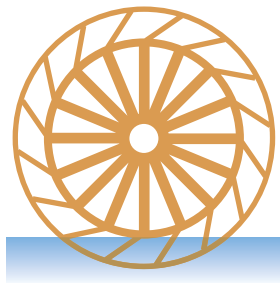
Set II

Piano Keyboard. Imagine that the keyboard of a piano extends endlessly in both directions.

29. Draw a picture of the part of the keyboard that repeats.
30. What type of symmetry does the "endless" keyboard have because of this repeating pattern?
29. *(continued)* Draw the lines on your figure through which the "endless" keyboard could be reflected so that it coincides with itself.
31. Is there any point about which the "endless" keyboard can be rotated less than 360° so that it coincides with itself? If so, where is it?
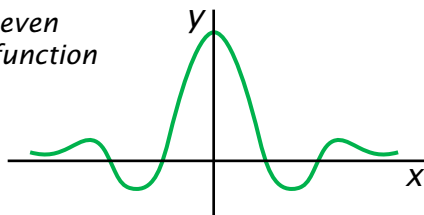
Water Wheel. The design of an old type of water wheel is shown here.*
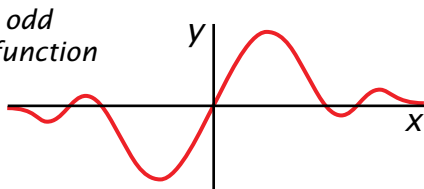
32. What types of symmetry does it have?

33. What is the measure of the smallest angle through which it can be turned to look exactly the same?

Would the wheel look exactly the same if it were turned

34. 125°?

35. 225°?

36. To say that the wheel has "*n*-fold" symmetry, what number should *n* be?

37. Does it have reflection symmetry?

Wave Functions. Elementary particles such as electrons and photons are associated with "wave functions." The two figures below are graphs of wave functions.†

An even wave function

An odd wave function

38. What is the axis of symmetry of the graph of the even function?

39. What type of symmetry does the graph of an even function have?

40. What is the center of symmetry of the graph of the odd function?

41. What type of symmetry does the graph of an odd function have?

What type of function–even, odd, or neither–is described by each of these statements?

42. If the point (a, b) is on its graph, then so is the point $(-a, b)$.

43. If the point (a, b) is on its graph, then so is the point $(a, -b)$.

44. If the point (a, b) is on its graph, then so is the point $(-a, -b)$.
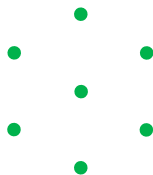
Cherry Orchard. Imagine that this orchard of cherry trees extends endlessly without any boundaries.‡

45. Why does the "infinite" orchard have translation symmetry?

46. What is the shortest distance related to the trees that the orchard can be translated and coincide with itself?

Cats' Paws and Catapults: Mechanical Worlds of Nature and People, by Steven Vogel (Norton, 1998).
†*Reality's Mirror: Exploring the Mathematics of Symmetry,* by Bryan Bunch (Wiley, 1989).
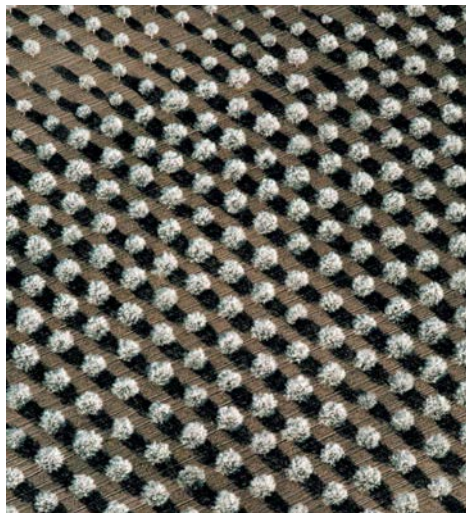
‡*Below from Above,* by Georg Gerster (Abbeville Press, 1986).

47.

Copy the figure above illustrating one tree and its six closest neighbors. Draw all of its lines of symmetry.

48. Why does the orchard have reflection symmetry?

49. Why does the orchard have rotation symmetry?

50. What points could serve as centers of symmetry?

51. What is the smallest angle through which the orchard can be turned to coincide with itself?

Set III

Short Story. In the following very short story by Scott Kim, some of the words have missing parts. In each case, the missing part is congruent to the part shown but rotated 180°. For example,

$\partial n \quad \rightarrow \partial nnie$

Scott Kim also created this figure.

2. What is unusual about it?

1. Copy the story and fill in the missing parts.

$\dot{u}$ the summer, $S^{\dot{1}}$ goes to the beach

and SU herself, adds up SU

of numbers, does $S\mp U$ and collects

ti cans. The ride home is hjl , and

that makes her head feel fiz .

Summary and Review

Basic Ideas

Summary of Key Ideas

Every isometry of the plane can be identified as a single reflection (a mirror image), as two reflections (a translation or rotation), or as three reflections (a glide reflection).

An isometry of the plane that changes a figure into its mirror image is either a reflection or a glide reflection. An isometry that does *not* change a figure into its mirror image is either a translation or a rotation.

Construction

8. To reflect a point through a line.   306

Exercises

Set I

Amazing Vase. This vase was created for Queen Elizabeth's Silver Jubilee in 1977.

1. Does this photograph appear to have reflection symmetry? Explain.
2. What is unusual about the vase?

Double Meanings. To an electrician, a *transformer* is a device used to transfer electric energy from one circuit to another.

3. What does the word *transformation* mean in geometry?
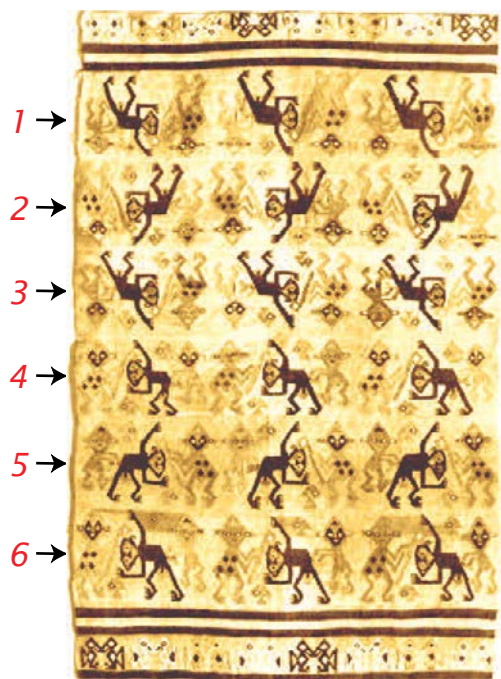
To an athlete, the word *isometric* refers to an exercise in which a muscle is tensed without changing its length.

4. What does the word *isometry* mean in geometry?

To an eye doctor, the word *dilation* refers to enlarging the pupil of the eye.

5. What does the word *dilation* mean in geometry?

6. Is a dilation an isometry?

Monkey Rug. This tapestry was woven in Peru sometime between 1000 and 1500 A.D.

1 →
2 →
3 →
4 →
5 →
6 →

The following questions refer to the rows (numbered in red) of dark brown monkeys. Name the transformation through which the

7. third row appears to be the image of the first row.

8. second row appears to be the image of the first row.

9. fourth row appears to be the image of the third row.

10. sixth row appears to be the image of the second row.

Clover Leaves. An ordinary clover leaf has 3-fold rotation symmetry.

11. What does "3-fold" mean?

12. What is the smallest angle through which the leaf can be turned to look the same?

13. Does an ordinary clover leaf have point symmetry? Explain.

14. Sketch the leaf and draw its lines of symmetry.

15. Sketch a clover leaf with four "leaves" and draw its lines of symmetry.

16. What is the smallest angle through which it can be turned to look the same?

17. Does this kind of clover leaf have point symmetry?

18. What kind of luck is associated with the clover leaf that you drew in exercise 15?

Musical Transformations. The first measure of a piece of music by Bach is shown here.*

19. What transformation appears in it?

Another composer, Arnold Schoenberg, wrote about taking a sequence of musical notes and transforming it in different ways.†
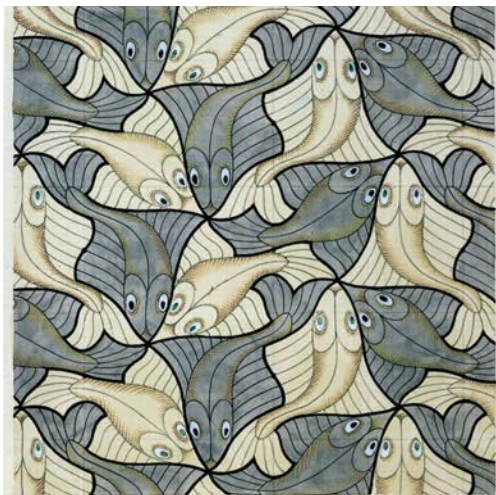
What geometric transformation corresponds to each of the following musical transformations?

20. Retrogression (playing a sequence of notes backward).

21. Transposition (moving a sequence of notes up or down).

22. Inversion (turning a sequence of notes upside down).

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*First Prelude (BWV 846) from *The Well-Tempered Clavier,* Vol. 1.
† *Emblems of Mind: The Inner Life of Music and Mathematics,* by Edward Rothstein (Times Books, 1995).
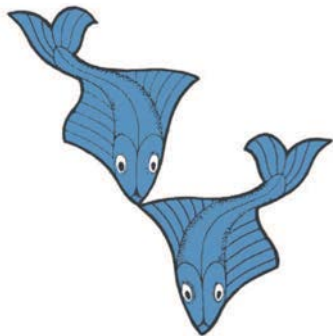
Fish Design. This is one of Escher's mosaics based on fish.
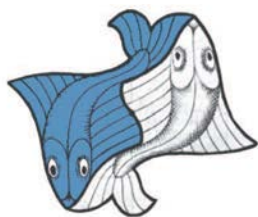
23. Do all of the fish in the figure appear to be congruent? Explain.

24. Define congruence with respect to isometry.

25.

Through what transformation is one of the fish in this part of the mosaic the image of the other?

26.
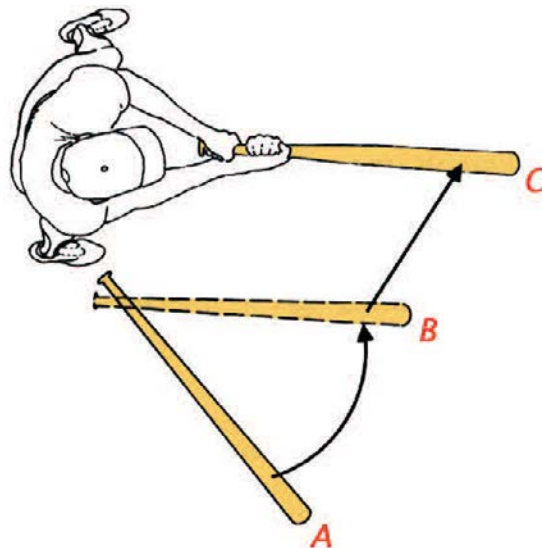
Through what transformation is one of the fish in this part of the mosaic the image of the other?

27. Do any other types of transformation appear in Escher's mosaic? If so, draw a figure to illustrate an example.

Set II

Batter's Swing. The swing of a baseball bat includes two types of motion.*
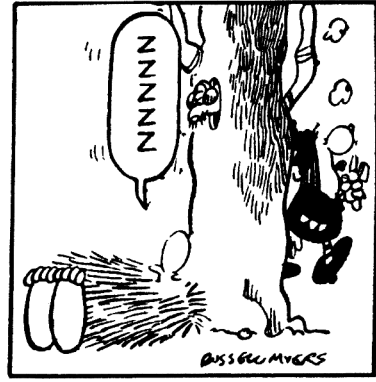
In the figure above, under what transformation is the bat

28. at B the image of the bat at A?

29. at C the image of the bat at B?

Each of these transformations can be considered the composite of two reflections.
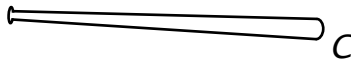
30. What is the difference in the way that the lines of reflection are related?

31. Use your protractor to estimate the magnitude of the rotation.

32. Given that the length of the bat is 42 inches, use the figure to estimate the magnitude of the translation. (Hint: *Measure everything in millimeters.*)

_____

Keep Your Eye on the Ball: The Science and Folklore of Baseball, by Robert G. Watts and A. Terry Bahill (W. H. Freeman and Company, 1990).
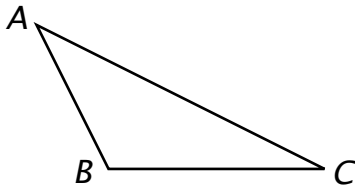
33. Trace the figure below of the bats in positions B and C and show, by an accurate drawing, one way in which the transformation can be obtained geometrically as the composite of two reflections.

C

B
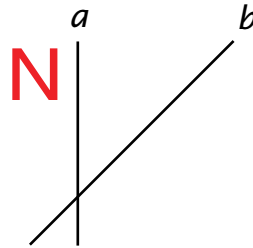
Construction Problem. △ABC has no special symmetry.

A

B   *C*

34. Draw a large copy of it on your paper. Use your straightedge and compass to construct the reflection of point B through line AC. Label its image D. Draw AD and CD.

35. What are the images of AB and BC through line AC?

36. Why is △ABC ≅ △ADC?

37. Describe the symmetry of quadrilateral ABCD.

34. *(continued)* Construct the reflection of point A through line DC. Label its image E. Draw DE and CE.

38. For what type of transformation is △EDC the image of △ABC?

From N to Z. Make an accurate copy of the following figure.

a   *b*

N

39. Sketch the reflection of the letter N through line *a*, and then sketch the reflection of the image that results through line *b*.

40. What transformation is the composite of these two reflections?

41. What does the measure of the acute angle formed by lines *a* and *b* appear to be?

42. What does the magnitude of the composite transformation appear to be?

43. Is there any other letter of the alphabet whose final image from reflections through these two lines is another letter of the alphabet? Explain.

Past and Future. The following figures are by graphic designer John Langdon.*
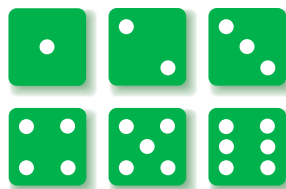
Think of each pattern in the figures as repeating endlessly in both directions.

44. Draw a picture of the part of each pattern that repeats.

45. What type of symmetry does each figure have owing to this repeating pattern?

44. *(continued)* Draw the lines on your figures through which the "endless" patterns could be reflected so that each coincides with itself.

46. Why have no vertical bars been drawn on the Es?

47. Does either pattern have rotation symmetry?

48. Which letters in the patterns have been drawn so that they have reflection symmetry?

49. Have any letters been drawn so that they have rotation symmetry? If so, which one(s)?

Dice Symmetries. The six faces of a die do not all have the same symmetry.

50. Sort them into groups in which the faces of each group have the same symmetry and describe the symmetries of each group.

_____

Wordplay, by John Langdon (Harcourt Brace Jovanovich, 1992).

Midterm Review

Set I

Chill Factor. A 3-year-old child can understand the statement "You can't go outside if you don't put your coat on."*

1. What type of statement is it?
2. What is its hypothesis?

Losers, Sleepers. The following questions are about the diagram at the right.

3. What is a diagram of this type called?
4. Write the statement represented by the diagram.
5. What is its conclusion?

Marine Logic. Sea lions that have been taught with symbols that $a \rightarrow b$ and $b \rightarrow c$ seem to realize that $a \rightarrow c$.†

6. What is an argument of the form

$$a \rightarrow b$$
$$b \rightarrow c$$
$$\text{Therefore, } a \rightarrow c$$

called?

7. What could cause the conclusion of such an argument to be false?

Finding Truth. G. K. Chesterton once wrote: "You can only find truth with logic if you have already found truth without it."

What are the statements called that we

8. prove to be true by using logic?
9. assume to be true without proof?

_____
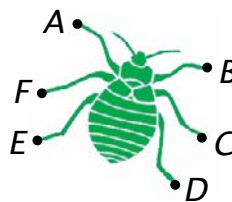
Why Do Buses Come in Threes? The Hidden Mathematics of Everyday Life, by Rob Eastaway and Jeremy Wyndham (Wiley, 1998).
†*The Animal Mind,* by James L. Gould and Carol Grant Gould (Scientific American Library, 1994).

Only in Geometry. Each of the following words is an important term used *only* in geometry. Briefly explain what each word means.

10. Collinear.
11. Hypotenuse.
12. Isosceles.
13. Rhombus.

Why Three? Most insects walk on three of their six legs at a time.

14. What postulate tells you that the insect in the figure above won't tip over when legs A, C, and E touch the ground?
15. Which geometric terms in this postulate are undefined?

What Follows? Complete the statements of the following postulates and theorems.
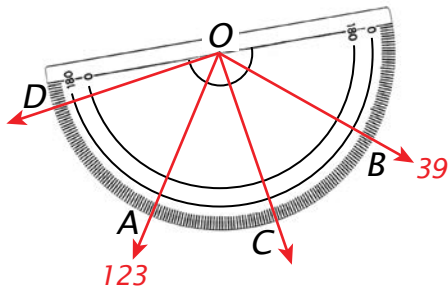
16. Two points determine . . .
17. The sum of the angles of a triangle is . . .
18. An angle has exactly one ray that . . .
19. Complements of the same angle . . .
20. An equilateral triangle is . . .
21. An exterior angle of a triangle is greater than . . .
22. In a plane, two points each equidistant from the endpoints of a line segment determine . . .
23. Equal corresponding angles mean that lines . . .
24. In a plane, two lines perpendicular to a third line . . .
25. In a plane, a line perpendicular to one of two parallel lines is . . .

26. An exterior angle of a triangle is equal to the sum of . . .

27. A quadrilateral is a parallelogram if its opposite angles are . . .

28. The diagonals of an isosceles trapezoid . . .

Formulas. Explain what each of the following formulas means.

29. $A = \pi r^2$.

30. $A = lw$.

31. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

32. $p = a + b + c$.

33. $p = 2l + 2w$.

Protractor Problems. The figure below shows a protractor placed on ∠AOB so that the coordinates of its sides are 123 and 39.
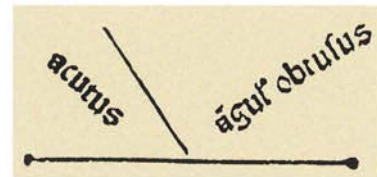
34. Find the measure of ∠AOB.

35. Find the measures of ∠AOC and ∠COB, given that ray OC bisects ∠AOB.

36. Find the coordinate of ray OC.

37. Find the measure of ∠AOD, given that ∠AOD and ∠AOC are complementary.

38. Find the coordinate of ray OD.

39. Would it be correct to say that ray OC is between ray OD and ray OB? Explain why or why not.

Metric Angles. As part of the metric system adopted at the time of the French revolution, right angles were defined to have a measure of 100 "grades."

Write each of the following definitions or theorems in terms of "grades."

40. The definition of supplementary angles.

41. The definition of an obtuse angle.

42. The theorem about the sum of the angles of a quadrilateral.

43. The theorem that each angle of an equilateral triangle is 60°.

Linear Pair. The figure below appeared on the opening page of the first printed edition of the *Elements* (1482).

44. What do you need to know about this figure to know that it contains a linear pair?

45. If two angles form a linear pair, does it follow that one is acute and the other is obtuse? Explain.

46. In what way are the measures of the angles of every linear pair related?

47. If the angles in a linear pair are equal, what can you conclude about each angle?
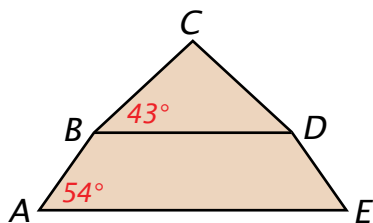
Polygons. In each of the following exercises, the measure of one or more parts of a polygon is given. Tell what the measures of the indicated parts must be.

48. A right triangle. The other side if the two shorter sides are 15 and 30.

49. An isosceles triangle. The third side if two sides are 15 and 30.

50. A rhombus. The other angles if one angle is 30°.

51. An isosceles trapezoid. The other angles if one angle is 30°.

Bent Pyramid. The Bent Pyramid at Dahshur in Egypt is shown in the photograph below.
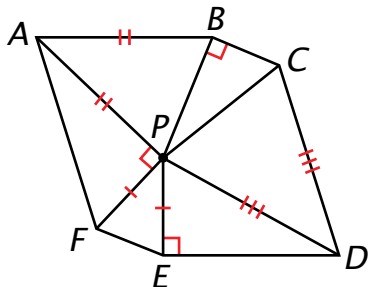
In the following side view of it, ABDE is an isosceles trapezoid and △BCD is an isosceles triangle.

Use the facts that ∠A = 54° and ∠CBD = 43° to find

52. ∠ABD.

53. ∠C.

54. ∠E.

55. ∠CDE.

Six Triangles. Three triangles in the following figure are right triangles, and the other three are isosceles.
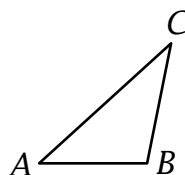
56. Two triangles look as if they might be congruent. Which two are they?

57. If you knew that AB = ED, you could prove these triangles congruent. How?

58. You could also prove them congruent if you knew that AF = CD. How?

Italian Theorem. The following theorem appears in an Italian geometry book.

Teorema. Se due lati di un triangolo sono disegauli, l'angolo opposto al lato maggiore è maggiore di quello opposto al lato minore.

In terms of the figure above, the hypothesis of this theorem might be stated as: "In △ABC, CA > CB."

59. State the conclusion in terms of the figure.

60. State the theorem as a complete sentence in English.

61. What English words are related in meaning to "maggiore" and "minore"?

Impossibly Obtuse. Recalling that an acute triangle has three acute angles, Obtuse Ollie concluded that an obtuse triangle has three obtuse angles.

The beginning of a proof that such a triangle is impossible follows.

Proof

Suppose that △ABC has three obtuse angles.
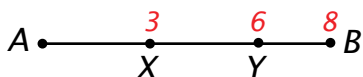
Then ∠A > 90°, ∠B > 90°, and ∠C > 90°.

62. What follows from adding these inequalities?

63. What does this result contradict?

64. What does the contradiction show about what we supposed?

65. What kind of proof is this proof?

Converses. Write the converses of the following statements and tell whether the converses are true or false.

66. If two angles are the acute angles of a right triangle, then they are complementary.

67. If a quadrilateral is a parallelogram, then the diagonals of the quadrilateral bisect each other.

Construction Exercises.

68. First, use your ruler to draw a line segment 8 cm long. Label its endpoints A and B and mark the points X and Y on it that are 3 cm and 6 cm from point A.
    Use your straightedge and compass to construct △ABC with AC = AY and BC = BX.
    Construct the line that bisects ∠A.

69. Does this line also bisect BC?

68. *(continued)* Check your answer by constructing a line that bisects BC.

68. *(continued)* Through C, construct a line perpendicular to AB. Let P be the point in which this line intersects AB.

70. What point on line AB is closest to point C?

71. Which three lines in the figure are concurrent?
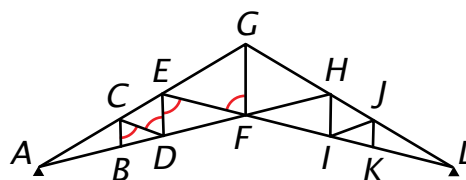
Set II

Grid Exercise. On graph paper, draw a pair of axes extending 20 units to the right and 10 units up from the origin.

72. Plot the following points and connect them with line segments to form quadrilateral ABCD: A(0, 2), B(12, 2), C(16, 8), D(4, 8).

73. What kind of quadrilateral does ABCD appear to be?

74. Find the lengths of its sides to check your answer.

72. *(continued)* Plot P(10, 2) and draw PD and PC.

75. Which segment looks longer: PD or PC?

76. Use the distance formula to check your answer.

Roof Truss. The figure below from a book on architecture illustrates one type of truss, a structure used to support a roof.

In the structure, GA = GL and FA = FL.

77. What can you conclude about ∠A and ∠L? Explain.

Also, ∠BCD = ∠CDE = ∠DEF = ∠EFG.

78. What can you conclude about BC and FG? Explain.
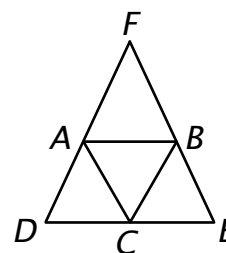
It appears as if CD ∥ EF.

79. Is this necessarily true? Explain.

∠AGF is acute and ∠GFA is obtuse.
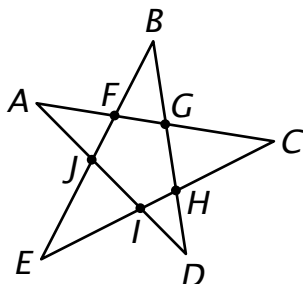
80. Can these two angles be supplementary? Explain.

Angles Problem 1. In the figure at the right, △ABC is equilateral and its vertices lie on the sides of △DEF, AB ∥ DE, ∠D = 65°, and ∠F = 50°.
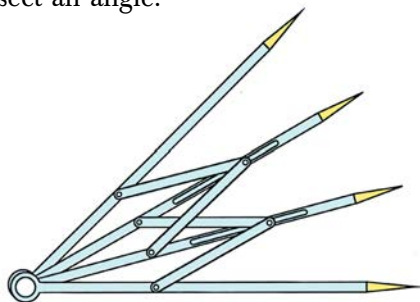
81. Make a large copy of the figure and write the given information on it. Find the measures of the other angles and write them on the figure.

82. What can you conclude about △ACD and △BCE? Explain.

Irregular Star. The five angles at the points of the star below, formed from five intersecting line segments, are equal.
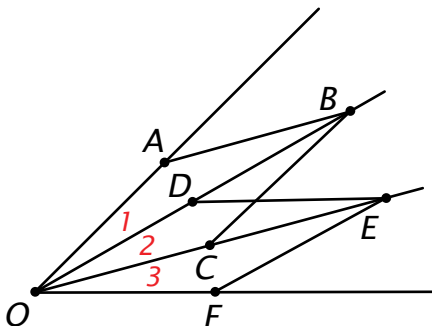
83. Copy the figure and mark it as needed to help in deciding whether each of the following statements is true. *If you think a statement is true, explain why.*

84. △EFC is isosceles.

85. Pentagon FGHIJ is equiangular.

86. Pentagon FGHIJ is equilateral.

87. Every triangle in the figure is isosceles.

Angle Trisector. The linkage below can be used to trisect an angle.
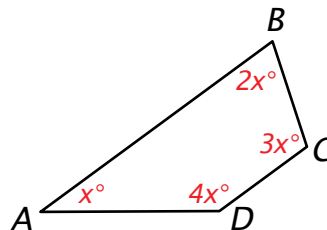
It is designed so that OA = AB = BC = CO = OD = DE = EF = FO.

Explain why each of the following statements is always true, no matter how the linkage is moved.
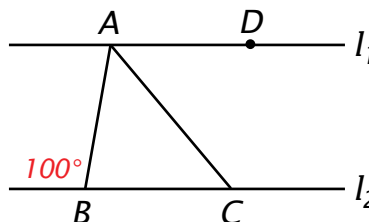
88. △OAB ≅ △OCB, and △ODE ≅ △OFE.

89. ∠1 = ∠2 = ∠3.

Quadrilateral Problem. The measures of the angles of quadrilateral ABCD are multiples of *x*.
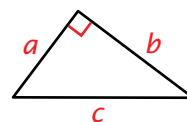
90. Write an appropriate equation and solve for *x*.

91. What is the measure of the largest angle of ABCD?

92. What other conclusions can you make about this figure?

Angles Problem 2. In the figure below, $l_1 \parallel l_2$ and AC bisects ∠BAD.
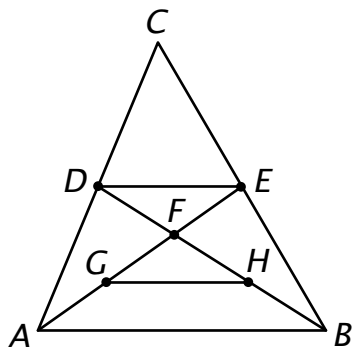
93. Make a large copy of the figure and write in the measure of the 100° angle. Find the measures of the other angles and write them on the figure.

94. What can you conclude about △ABC? Explain.

A, B, C. Suppose that the legs of a right triangle have lengths *a* and *b* and the hypotenuse has length *c*.
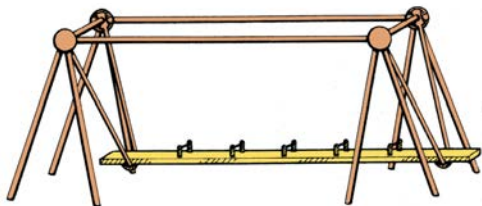
95. Write an equation relating a, b, and c.

96. Write three inequalities relating a, b, and c.

97. Could a triangle have sides of lengths a^2, b^2, and c^2? Explain.

Midsegments. In the figure below, D and E are the midpoints of AC and BC, AE and BD intersect at F, and G and H are the midpoints of AF and BF.

98. Copy the figure and mark it as needed to explain why each of the following statements is true.

99. DE = GH.

100. DE ∥ GH.

98. *(continued)* Draw DG and EH.

101. DEHG is a parallelogram.

102. AG = GF = FE and BH = HF = FD.

On the Level. The plank of this swing always stays level with the ground as the plank moves back and forth.*
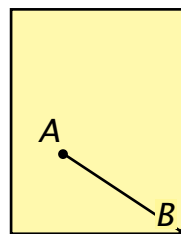
103. What is it about the design of the swing that makes the plank stay level?

*Mathematics Meets Technology, by Brian Bolt (Cambridge University Press, 1991).

Folding Experiment.
Mark a point on a sheet of paper and label the point A. Choose a corner of the paper, label it B, and draw segment AB. Fold the paper so that B falls on A, and make a sharp crease. Unfold the paper so that it is flat.
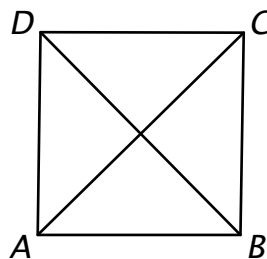
104. Draw a sketch of the result.

105. What relation does the crease line seem to have to AB?

106. Add some extra line segments as needed to help in explaining why.

Earth Measurement. In the ninth century, a team of surveyors working near Baghdad found that 1 degree at the center of the earth corresponded to about 57 miles on the surface.

With the use of this information, what did they estimate for

107. the circumference of the earth?

108. the distance from the surface of the earth to its center?

Not a Square. Although parallelogram ABCD looks like a square, it is not because ∠DAB ≠ ∠ABC.
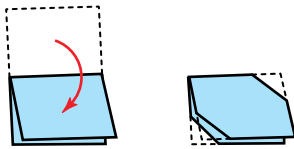
On the basis of this information, *could* it be that

109. ABCD is a rhombus? Explain.

110. AC ⊥ DB? Explain.

111. AC and BD bisect each other? Explain.

112. ABCD is a rectangle? Explain.

113. AC = DB? Explain.

Construction Exercise.
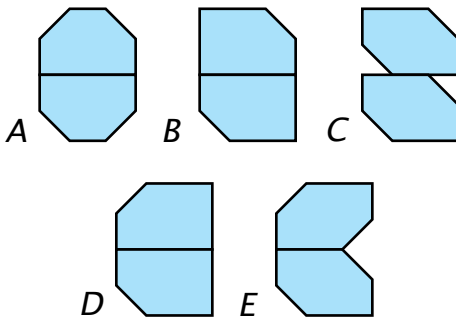
114. Draw a scalene triangle in the center of a sheet of paper and label its vertices A, B, and C. Use your straightedge and compass to construct three equilateral triangles outwardly on the sides of △ABC. Label the equilateral triangles △ABZ, △BCX, and △ACY.

115. Draw AX, BY, and CZ. What seems to be true about these three line segments?

SAT Problem. The figures below appeared in a problem on an SAT test.

The first figure shows a rectangular sheet of paper being folded in half. The second figure shows the result of cutting off two opposite corners of the folded paper.

In figure A below, the lower half is the image of the upper half by a reflection.
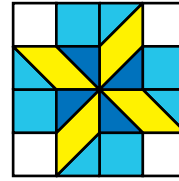
Name the transformation(s) through which the lower half is the image of the upper half in
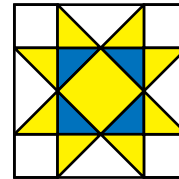
116. figure B.

117. figure C.

118. figure D.

119. figure E.

120. Which one of the five figures represents the paper when it is unfolded?

Quilt Patterns. Three patterns used in making quilts are shown below.* Describe the symmetries of each pattern.
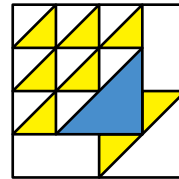
121.
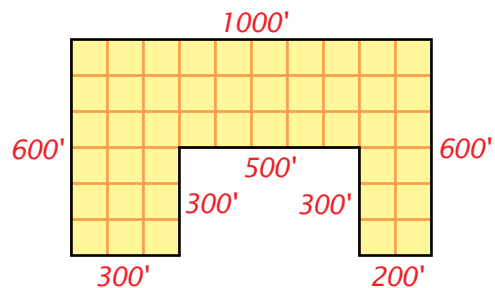
122.

123.

Dividing a Lot. A real estate developer[†] is planning to subdivide a vacant lot into congruent parcels (the ′ stands for feet).
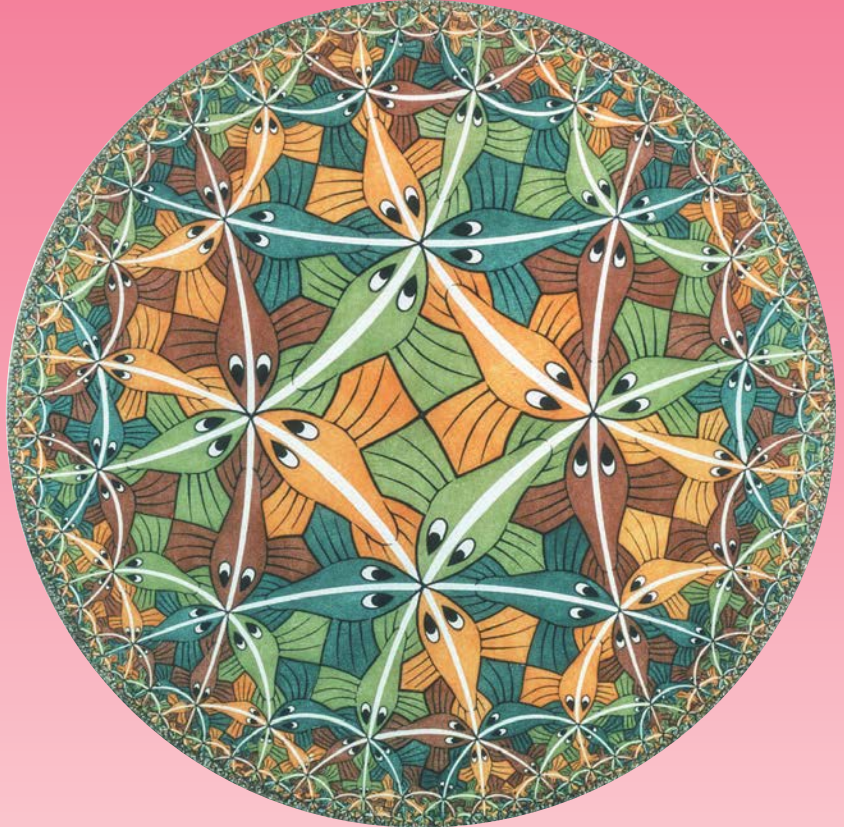
Make a sketch to illustrate each of the following divisions.

124. A way to divide the lot into three congruent parcels.

125. A way to divide it into five congruent parcels.

Designing Tessellations, by Jinny Beyer (Contemporary Books, 1999).
[†]Solomon Golomb.
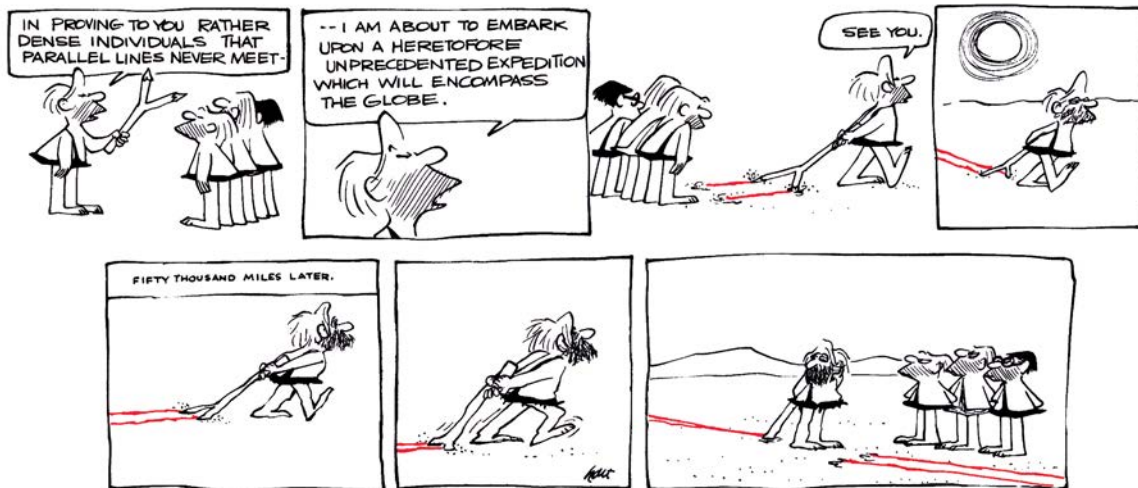
Chapter 16

Non-Euclidean Geometries

A remarkable thing happened in the nineteenth century. Mathematicians realized for the first time that there are geometries other than that of Euclid. These geometries are based on different assumptions and lead to theorems that seem to contradict common sense. Nevertheless, these geometries, called non-Euclidean, are logically consistent, and one of them may actually be the correct system for describing our universe!

Used by permission of Johnny Hart and Creators Syndicate, Inc.
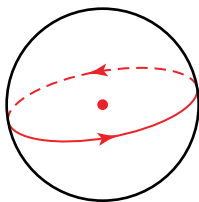
Geometry on a Sphere

Euclid defined parallel lines as "lines that, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction." In the cartoon above, one of the characters tries to prove that parallel lines never meet by tracing them on the surface of the earth. His two-pronged stick, however, has worn down to a nub by the end of the journey, and so the "parallel lines" have come together to meet in a common point.

Aside from the wearing down of the stick, the cartoon shows a way in which geometry on a sphere differs from the plane geometry that we have been studying. We think of lines as being *straight* and *infinitely long,* but "lines" drawn on the surface of a sphere are *curved* and *finite in extent.* The "plane" on which they are drawn, the surface of the sphere, is not flat but curved and is finite in extent.
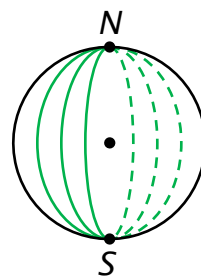
Peter, the character in the cartoon who tried to prove that parallel lines never meet by tracing them around the earth, has come back to his starting point. This suggests that he walked "straight ahead," never turning either left or right. Such a path is along a *great circle* of the sphere.

Definition
A *great circle* of a sphere is a set of points that is the intersection of the sphere and a plane containing its center.

From this definition, it follows that a great circle divides the surface of a sphere into two equal parts, called *hemispheres*. The most famous great circle on the earth is the equator, which divides the earth into the northern and southern hemispheres. Other well-known great circles on the earth are the meridians, the circles that pass through the North and South Poles. The Poles, where the axis about which the earth turns intersects the earth's surface, are an example of a pair of *antipodal points*.
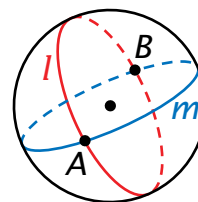
Definition
Antipodal points are the two points of intersection of a sphere with a line through its center.

At the beginning of our study of geometry, we did not define the terms "point," "line," and "plane" but gave them meaning by making postulates about them. Would these postulates still seem reasonable if we changed our ideas of these terms? For example, suppose that, in thinking about geometry on a sphere, we call great circles "lines" and the surface of the sphere a "plane."
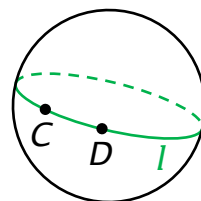
Our first postulate said:

Two points determine a line.

If great circles are "lines," the figure at the right suggests that this postulate is no longer true. Lines *l* and *m* intersect in two antipodal points, A and B; so points A and B evidently do *not* determine a line.
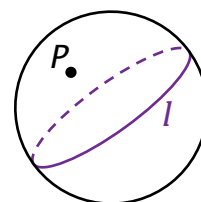
Suppose, however, that we consider a pair of antipodal points to be just one "point." (Remember that, as with "line" and "plane," we did not define the term "point".) If we do consider a pair of antipodal points to be just one "point," then our first postulate is true again. Lines *l* and *m* intersect in just one point (called A and B). To determine a line, we need two points—C and D in the next figure at the right.

With our new ideas about points and lines comes a surprising result. The last figure on this page represents a point, P, and a line, *l*, that does not contain it. Through P, how many lines can be drawn parallel to line *l*? The answer is *none. Every* great circle of a sphere intersects all other great circles of the sphere!

What does all this mean? It means that we have the beginning of a geometry in which the Parallel Postulate that we have used in the past no longer applies, because, through a point not on a line, there are now *no lines parallel to the line.*

With other changes in our postulates concerning distance and betweenness and this new assumption that there are no parallel lines, we can develop a new geometry with all sorts of unexpected theorems. Although these theorems flatly contradict others that we have proved in our study of geometry, they make sense in regard to the new parallel postulate and in regard to one another. They are part of a *non-Euclidean* geometry. In this chapter, we will become acquainted with the two main non-Euclidean geometries.

Exercises

In the following exercises, we will refer to the geometry that we have been studying throughout the course as *Euclidean* geometry and to our new geometry on a sphere as *sphere* geometry. The surface that we use for our model of sphere geometry is curved in three-dimensional space, but the surface itself is two-dimensional and we will think of it as a "plane."

Set I

Garden Sphere. This beautiful sculpture of a sphere consists of polished stainless steel hoops to form intersecting *great circles*.
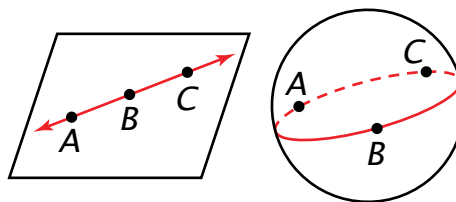
MOORE DESIGNS

1. Where is the center of a great circle of a sphere?
2. What is its radius?
3. Into what does a great circle divide a sphere?

Any two great circles of a sphere intersect in a pair of *antipodal points*.

4. What type of line segment inside a sphere connects two antipodal points?
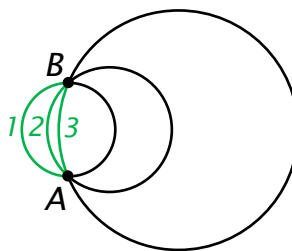
Basic Differences. The figures below illustrate some simple differences between Euclidean geometry and sphere geometry. Each figure represents three points and a line that contains them.*

Tell whether each of the following statements seems to be true for one of the geometries (name it) or both of them.

5. Points A, B, and C are collinear.
6. Point B is between points A and C.
7. If AB = BC, then B is the midpoint of AC.
8. AB = BC = AC.
9. AB + BC = AC.
10. If two points lie in a plane, the line that contains them lies in the plane.
11. Point B separates the line into two parts.

Shortest Routes. The figure below from Euclidean geometry shows two points connected by arcs of three circles.
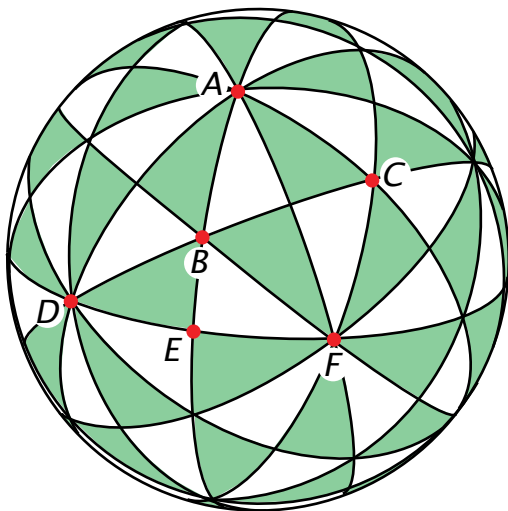
12. Which of the three numbered arcs is the shortest?
13. How does the circle of which the arc is part compare in size with the other two circles?

———————
*We take the liberty of using some of the words of Euclidean geometry in sphere geometry without formal definitions when the meaning is clear by analogy or from examples given.

14. From these results, how would you describe the shortest path from A to B in sphere geometry?

15. Why is this part of sphere geometry of interest to the airlines?

Beach Ball. This beach ball has been divided by great circles into congruent triangles.

Each triangle is a 36°-60° right triangle.

16. What is the sum of the angles of one of these triangles?

17. Are the acute angles of a right triangle in sphere geometry complementary? Explain.
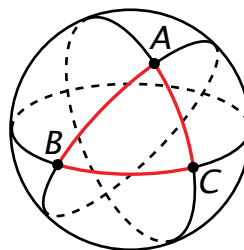
Find the measures of the angles in each of the following triangles and then find their sums.

18. △ABC.

19. △AEF.

20. △CDF.

21. △ADF.

22. If a triangle in sphere geometry is equiangular, does it also appear to be equilateral?

23. How does the sum of the angles of a triangle in sphere geometry seem to be related to its area?

24. What kind of triangle would you expect to have an angle sum very close to 180°?

Set II

Spherical Triangles. Menelaus, a Greek mathematician, wrote a book on sphere geometry in about 100 A.D. In it, he proved many theorems about spherical triangles.

25. What figures are the sides of a triangle in sphere geometry?

Menelaus proved that the Triangle Inequality Theorem of Euclidean geometry is also true in sphere geometry.

26. What does this theorem say?

He proved that, if the angles of one triangle are equal to the angles of another triangle, then the triangles are congruent.
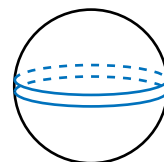
27. Is this statement true in Euclidean geometry? Explain.

Menelaus also proved that the sum of the angles of a triangle in sphere geometry is related to the area of the triangle.
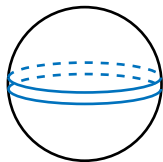
28. What do you think the connection is?

Seemingly Parallel. The figure below seems to show two lines in sphere geometry that are parallel.

29. Why do the two curves seem to be parallel?
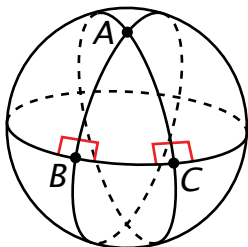
Parallel lines do not exist on a sphere; so something is wrong with our thinking about this figure.

30. What is it?

Euclidean and Sphere Geometries. In the figure below, lines AB and AC are both perpendicular to line BC.

31. In Euclidean geometry, how many lines through a given point in a plane can be perpendicular to a given line in the plane?

32. Is your answer to exercise 31 also true in sphere geometry?

33. In Euclidean geometry, what can be concluded about two lines that form equal corresponding angles with a transversal?

34. Is this conclusion also true in sphere geometry? Explain.

35. In Euclidean geometry, what can be concluded about two lines that lie in a plane and are perpendicular to a third line?

36. Is this conclusion also true in sphere geometry?

37. In Euclidean geometry, what can be concluded about an exterior angle of a triangle with respect to the remote interior angles?

38. Is this conclusion also true in sphere geometry? Explain.

39. In Euclidean geometry, what can be concluded about the sum of the measures of the angles of a triangle?

40. Is this conclusion also true in sphere geometry? Explain.

Wet Paint. Obtuse Ollie's father painted a straight line on the floor of his garage. While the paint was still wet, Ollie rolled his father's bowling ball along the line.
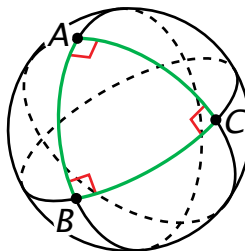
41. In what ways is a line painted on the floor different from a line in Euclidean geometry?

Acute Alice told Ollie that the ball had a line on it.

42. Does Alice's statement make any sense? Explain.

43. What determines the length of a line on a sphere?

Equilateral and Right. A triangle of the type shown in the figure below cannot exist in Euclidean geometry. △ABC is an equilateral right triangle!

In what way is it different from triangles in Euclidean geometry that are
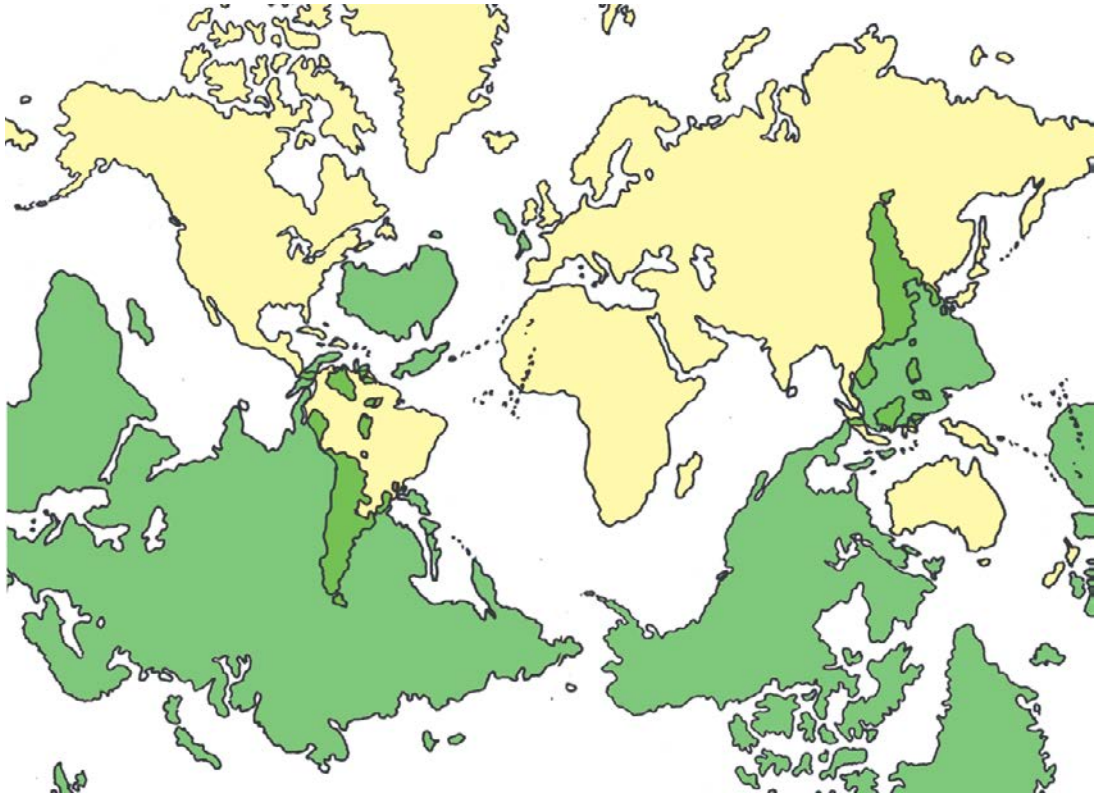
44. equilateral?

45. right?

46. How does the line through B and C compare in length with the sides of △ABC?

Given that the radius of the sphere is 2 units, what is
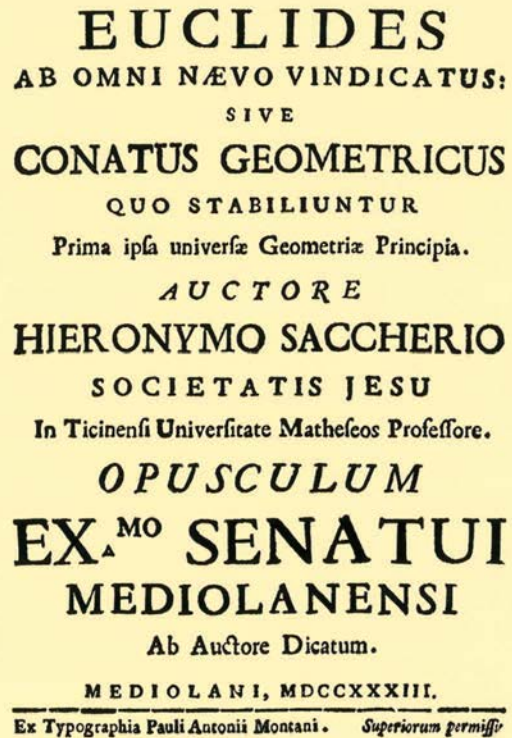
47. the length of the line through B and C?

48. the perimeter of △ABC?

49. the area of the sphere?

50. the area of △ABC?

Set III

Antipodal Points. If you could dig a hole through the center of the earth from where you are to the opposite side, where would you end up? One way to find out would be to turn a world globe so that your location is at the top and then look at the bottom. Another way would be to use the map below.

1. What do you think the map represents?

2. What does the map show that is rather surprising?

EUCLIDES

AB OMNI NÆVO VINDICATUS:

SIVE

CONATUS GEOMETRICUS

QUO STABILIUNTUR

Prima ipfa univerfæ Geometriæ Principia.

AUCTORE

HIERONYMO SACCHERIO

SOCIETATIS JESU

In Ticinenfi Univerfitate Matheſeos Profeſſore.

OPUSCULUM

EX.ᴹᴼ SENATUI

MEDIOLANENSI

Ab Auctore Dicatum.

MEDIOLANI, MDCCXXXIII.

Ex Typographia Pauli Antonii Montani.   Superiorum permiſſu
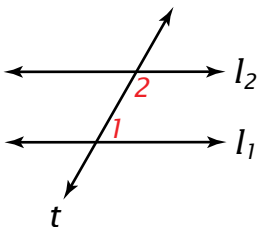
LESSON 2

The Saccheri Quadrilateral

The most controversial statement in the history of mathematics was made by Euclid near the beginning of the *Elements*. After presenting a series of definitions, Euclid listed five postulates, the last of which said:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

In regard to the figure at the left, this postulate says that, if $\angle 1 + \angle 2 < 180°$, then l_1 and l_2 must eventually intersect in a point on the right side of t.

Euclid's Fifth Postulate was criticized almost immediately, not only because it was much more complicated than the first four postulates, but also because many felt that he could have proved it on the basis of the first four.

A briefer statement that implies Euclid's Fifth Postulate is our Parallel Postulate:

> Through a point not on a line, there is exactly one line parallel to the given line.

Among the many people who tried to prove Euclid's Fifth Postulate was an eighteenth-century Italian priest named Girolamo Saccheri. A professor of both philosophy and mathematics, Saccheri tried to use indirect reasoning to establish the postulate as a theorem. As you know, indirect reasoning begins by assuming that the opposite of what you want to prove is true and continues by showing that this assumption leads to a contradiction. In 1733, about 2,000 years after Euclid wrote the *Elements*, Saccheri wrote a book titled *Euclid Freed of Every Flaw.* He intended to prove that Euclidean geometry is the only logically consistent geometry possible.

He began his work with a quadrilateral that has a pair of sides perpendicular to a third side. We will call such a quadrilateral "birectangular."

Definition

A *birectangular quadrilateral* is a quadrilateral that has two sides perpendicular to a third side.

The two sides perpendicular to the third side are the *legs*, the side to which they are perpendicular is the *base*, and the side opposite the base is the *summit*. In the first figure at the right, ∠A and ∠B are the *base angles* and ∠C and ∠D are the *summit angles* of quadrilateral ABCD. Also, ∠C is the summit angle opposite leg AD, and ∠D is the summit angle opposite leg BC.
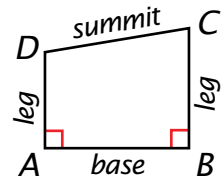
A birectangular quadrilateral whose legs are equal might be called "isosceles," but it is usually called a *Saccheri quadrilateral* in honor of Saccheri.

Definition

A *Saccheri quadrilateral* is a birectangular quadrilateral whose legs are equal.

A Saccheri quadrilateral looks very much like a rectangle; that is, the summit angles look as if they must also be right angles, from which it follows that the figure is equiangular. It is easy to prove that this is so in Euclidean geometry. In a geometry with a different postulate about parallel lines, however, it can be proved that a Saccheri quadrilateral is *not* a rectangle, because its summit angles are not right angles!

Before we look into these ideas further, we will state some theorems about birectangular quadrilaterals that are true in both Euclidean and non-Euclidean geometries. Each of them can be proved without using the Parallel Postulate, and their proofs are considered in the exercises. In the next lesson, we will use these theorems to derive some strange results that Saccheri obtained.

A Saccheri quadrilateral

Theorem 87

The summit angles of a Saccheri quadrilateral are equal.

Theorem 88

The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.

The next two theorems are comparable to a pair of theorems about inequalities in triangles that you already know.

Theorem 89

If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.

Theorem 90

If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

Exercises

Set I
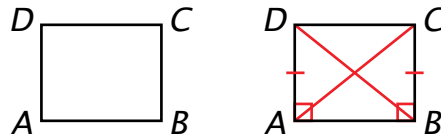
Wyoming. The boundary of Wyoming mapped on a plane can be thought of as a Saccheri quadrilateral whose base, YO, is its southern border.

1. What are sides WY and MO called with respect to quadrilateral WYOM?

2. What are ∠Y and ∠O called?

3. What is WM called?

4. What are ∠W and ∠M called?

5. Which sides of WYOM are equal?

6. Which angles of WYOM are right angles?

Complete the proofs of the following theorems of this lesson by giving the reasons.

Theorem 87. The summit angles of a Saccheri quadrilateral are equal.
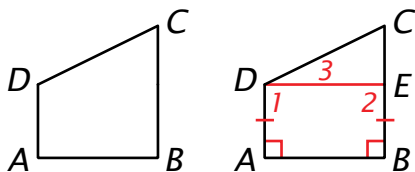
Given: Saccheri quadrilateral ABCD with base AB.
Prove: ∠C = ∠D.

Proof
7. Draw AC and BD. Why?

8. AD = BC. Why?

9. ∠BAD = ∠ABC. Why?

10. △BAD ≅ △ABC. Why?

11. BD = AC. Why?

12. △BCD ≅ △ADC. Why?

13. ∠BCD = ∠ADC. Why?

Theorem 89. If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.

Given: Birectangular quadrilateral ABCD with base AB, CB > DA.
Prove: ∠D > ∠C.

Proof
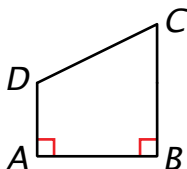
Because ABCD is a birectangular quadrilateral with base AB, DA ⊥ AB and CB ⊥ AB. Also, CB > DA.

14. Choose point E on BC so that BE = AD. Why?

15. Draw DE. Why?

16. ABED is a Saccheri quadrilateral. Why?

17. ∠1 = ∠2. Why?

18. Because ∠ADC = ∠1 + ∠3, ∠ADC > ∠1. Why?

19. ∠ADC > ∠2. Why?

20. ∠2 > ∠C. Why?

21. ∠ADC > ∠C. Why?

Theorem 90. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

Given: Birectangular quadrilateral ABCD with base AB, ∠D > ∠C.
Prove: CB > DA.

Proof (by the indirect method)

22. Either CB > DA, CB = DA, or CB < DA. Why?

Suppose CB = DA.
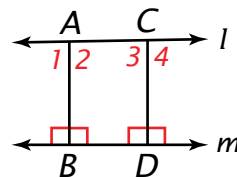
23. If CB = DA, what kind of quadrilateral is ABCD?

24. If your answer to exercise 23 is true, then ∠D = ∠C. Why?

25. What does this conclusion contradict?

Suppose CB < DA.

26. If CB < DA, then ∠D < ∠C. Why?

27. What does this conclusion contradict?

28. What is the only remaining conclusion about CB and DA?

Set II

Nasir Eddin. One of the many people who have tried to prove Euclid's Fifth Postulate was a Persian mathematician who was the court astronomer of the grandson of the famous Genghis Khan. His name was Nasir Eddin and he lived in the thirteenth century.
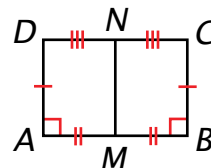
Nasir Eddin began by supposing that *l* and *m* are two lines such that perpendiculars from A and C to line *m* make ∠1 ≠ ∠2 and ∠3 ≠ ∠4.

29. If ∠1 and ∠3 are acute and ∠2 and ∠4 are obtuse, which segment must be longer: AB or CD?

30. Why?

Complete the following proof of Theorem 88 by giving the reasons.

Theorem 88. The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.
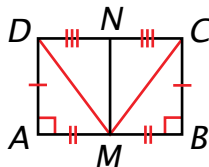
Given: Saccheri quadrilateral ABCD with MN connecting the midpoints of AB and CD.
Prove: MN ⊥ AB and MN ⊥ DC.

Proof
First, draw DM and CM.
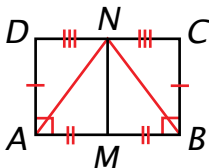
31. How can you conclude that DM = CM?

32. DN = CN. Why?

33. It follows that MN ⊥ DC. Why?
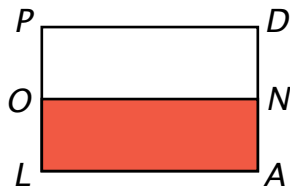
Next, draw AN and BN.

34. How can you conclude that AN = BN?

35. It follows that MN ⊥ AB. Why?

Polish Flag. The flag of Poland is shown in the figure below.

36. Copy the figure and mark the following information on it: PD = LA, ∠P and ∠L are right angles, O is the midpoint of PL, and N is the midpoint of DA.

Assuming *only* the information that you have marked on the figure and nothing about parallel lines, answer each of the following questions.

37. What kind of quadrilateral is PLAD?
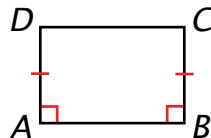
38. Which side is its summit?

39. Why is ∠D = ∠A?

40. Why is ON ⊥ PL and ON ⊥ DA?

41. What kind of quadrilaterals are OPDN and LONA?

Alter Ego. In Euclidean geometry, it is easy to prove that a Saccheri quadrilateral is a rectangle.

42. In Euclidean geometry, why would the legs of a Saccheri quadrilateral have to be parallel?
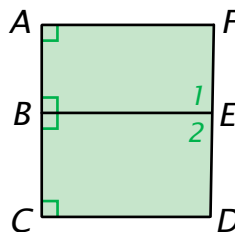
43. Why does it follow that a Saccheri quadrilateral is a parallelogram in Euclidean geometry?

44. Why does it follow that its summit angles are right angles?

45. Why does it follow that it is a rectangle?

46. In Euclidean geometry, what relations does the summit of a Saccheri quadrilateral have to its base?

Irregular Lots. The figure below shows an overhead view of two slightly irregular lots.

The angles at A, B, and C are right angles but the angles at D, E, and F are not; ∠1 > ∠F and ∠D > ∠2.

47. What can you conclude about the relative lengths of AF and CD? Explain.

48. If A-B-C and F-E-D, what can you conclude about the relative sizes of ∠F and ∠D? Explain.

Set III

Offsets. The jog in this road in Castleton, North Dakota, is caused by a basic problem in geometry.*
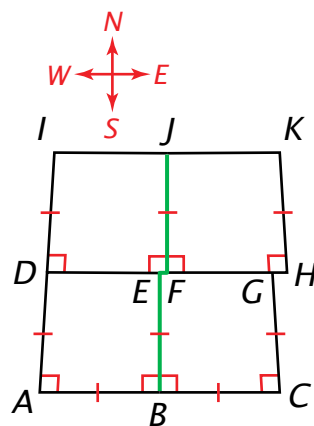
The figure at the right illustrates what the problem is. Each quadrilateral represents a "township," a unit of land used by surveyors. A township is supposedly bounded by a square measuring 6 miles on each side. If its sides lie along map directions as in the diagram, a township is effectively bounded by a Saccheri quadrilateral. The road from B to J has a jog in it from E to F.
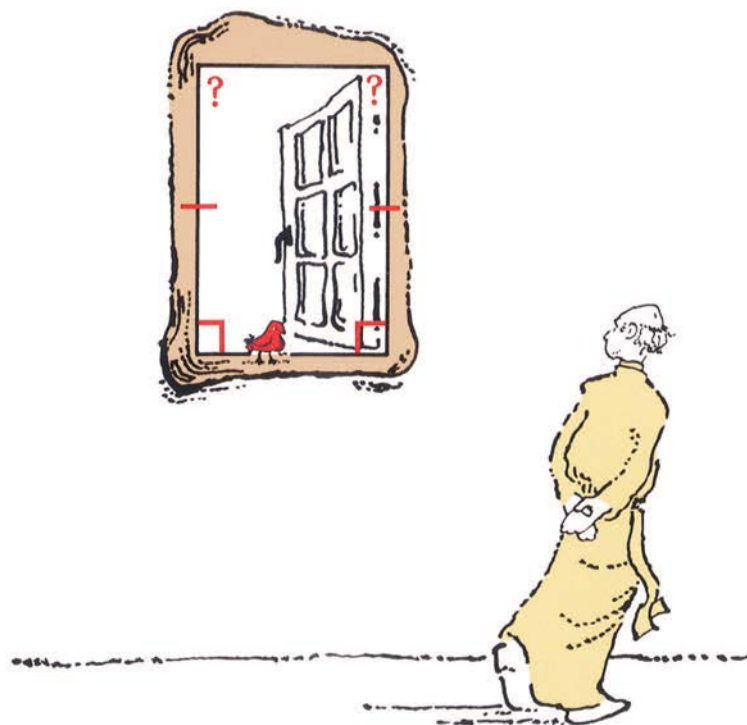
1. Are the legs of these Saccheri quadrilaterals parallel? Explain why or why not.

Each side marked with a tick mark is 6 miles long, and DF = FH = 6 miles.

2. How do you think the summits of these quadrilaterals compare in length with their bases? Explain.

3. How do you think the summits change as the townships continue farther north?

***Taking Measures Across the American Landscape*, by James Corner and Alex S. MacLean (Yale University Press, 1996).

The Geometries of Lobachevsky and Riemann

Saccheri thought that he could use his quadrilaterals to prove the Parallel Postulate. Having shown that the summit angles are equal, he planned to prove indirectly that they must also be right angles. To do so, he had to eliminate the possibilities of them being either acute or obtuse.

Saccheri managed to arrive at a contradiction by assuming that the summit angles are obtuse. Rather than being able to eliminate the possibility that they are acute, however, he ended up creating the beginning of a geometry that is non-Euclidean, that is, a geometry with a different assumption about parallel lines. Saccheri missed the implications of what he had started, and so he named his book *Euclid Freed of Every Flaw.* A better name would have been *A New Kind of Geometry.*

Saccheri died in 1733, a few months after his book was published. It was nearly a century later before anyone began to realize that geometries different from Euclid's made logical sense. Three men independently reached this conclusion: the great German mathematician Carl Friedrich Gauss; a Hungarian, Janos Bolyai; and a Russian, Nicolai Lobachevsky.

The assumption that Saccheri couldn't disprove–that is, that the summit angles of a Saccheri quadrilateral are acute–is the "acute angle" hypothesis. The non-Euclidean geometry based on this hypothesis is often called ***Lobachevskian geometry***, the name that we will use.

The Lobachevskian Postulate
The summit angles of a Saccheri quadrilateral are acute.

With the use of this postulate, the following theorems can be proved in Lobachevskian geometry.

Lobachevskian Theorem 1
The summit of a Saccheri quadrilateral is longer than its base.

Lobachevskian Theorem 2
A midsegment of a triangle is less than half as long as the third side.

Nicolai Lobachevsky
(1793–1856)

These theorems are false in Euclidean geometry. They contradict the theorems of Euclid that are *based on the Parallel Postulate*. The Parallel Postulate states that, through a point not on a line, there is exactly one line parallel to the given line. For the theorems stated above to hold in Lobachevskian geometry, it must be true that, through a point not on a line, there are many lines parallel to the line!

In Lesson 1, you learned that, in sphere geometry, there are *no* parallels to a line through a point not on it. However, this result contradicts other Euclidean theorems proved *before* the introduction of the Parallel Postulate. In fact, it is equivalent to assuming that the summit angles of a Saccheri quadrilateral are *obtuse*, which is why Saccheri was able to eliminate this possibility.

Nevertheless, if some other postulates related to distance and betweenness are changed in addition to the Parallel Postulate, a logically consistent non-Euclidean geometry can be developed in which there are no parallels at all. A German mathematician, Bernhard Riemann, was the first to understand this concept. We will refer to the geometry that he created as ***Riemannian geometry***.

The basic differences between Euclidean geometry and these two non-Euclidean geometries are summarized in the table below. In each case, either statement can be proved to be a logical consequence of the other.

Bernhard Riemann
(1826–1866)

Statement	*Euclid*	*Lobachevsky*	*Riemann*
Through a point not on a line, there is	*exactly one line* parallel to the line.	*more than one line* parallel to the line.	*no line* parallel to the line.
The summit angles of a Saccheri quadrilateral are	*right.*	*acute.*	*obtuse.*
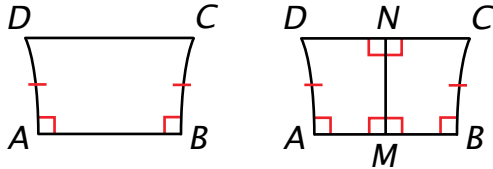
Exercises

We restrict our proofs to Lobachevskian geometry because, in this geometry, only the Parallel Postulate is changed. For this reason you can use any idea considered before Chapter 6 in this book.

Set I

Complete the proofs of the theorems of this lesson by giving the reasons. (Some of the sides of the figures have been drawn as curves to make the figures look non-Euclidean.)

Lobachevskian Theorem 1. The summit of a Saccheri quadrilateral is longer than its base.

Given: Saccheri quadrilateral ABCD with base AB.
Prove: DC > AB.

Proof
Because ABCD is a Saccheri quadrilateral with base AB, DA ⊥ AB and CB ⊥ AB. Therefore, ∠A and ∠B are right angles, and so ∠A = ∠B = 90°.

1. ∠D and ∠C are acute. Why?

2. ∠D < 90° and ∠C < 90°. Why?

3. ∠D < ∠A and ∠C < ∠B. Why?
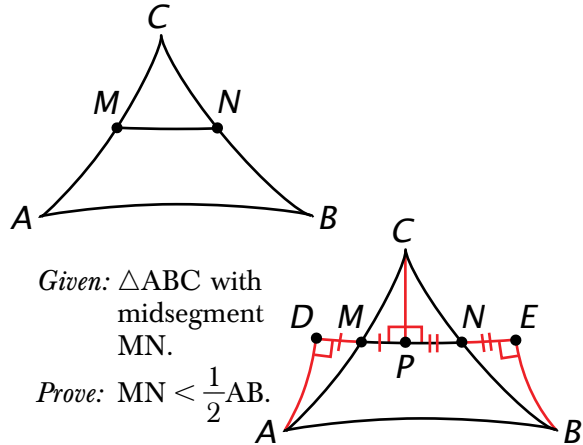
Let M and N be the midpoints of AB and DC as shown in the second figure.

4. What permits us to let M and N be these midpoints?

5. What permits us to draw MN?

6. MN ⊥ AB and MN ⊥ DC. Why?

7. AMND and BMNC are birectangular quadrilaterals. Why?

We have proved that ∠D < ∠A and ∠C < ∠B.

8. It follows that DN > AM and NC > MB. Why?

9. DN + NC > AM + MB. Why?

10. DC > AB. Why?

Lobachevskian Theorem 2. A midsegment of a triangle is less than half as long as the third side.

Given: △ABC with midsegment MN.
Prove: $MN < \frac{1}{2}AB$.

Proof
Through C, draw CP ⊥ MN. Choose points D and E on line MN so that MD = MP and NE = NP. Draw AD and BE.

11. △ADM ≅ △CPM and △BEN ≅ △CPN. Why?

12. AD = CP and BE = CP. Why?

13. AD = BE. Why?

Angles D and E are right angles because they are equal to the right angles at P; so AD ⊥ DE and BE ⊥ DE.

14. ADEB is a Saccheri quadrilateral. Why?

15. AB > DE. Why?

Because DE = DM + MP + PN + NE, MP = DM, and PN = NE, it follows that DE = MP + MP + PN + PN = 2MP + 2PN = 2(MP + PN).

16. DE = 2MN. Why?

17. AB > 2MN. Why?

18. $\frac{1}{2}AB > MN$, so $MN < \frac{1}{2}AB$. Why?

Set II

Lobachevsky and His Geometry. This stamp is one of two Russian stamps issued in honor of Nicolai Lobachevsky.
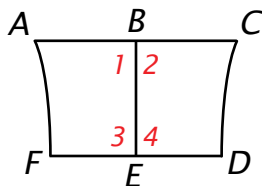
19. Can you guess what any of the words on the stamp mean?

Rewrite each of the following sentences from Euclidean geometry so that it is true in Lobachevskian geometry.

20. Through a point not on a line, there is exactly one line parallel to the line.

21. The summit angles of a Saccheri quadrilateral are right.

22. The summit of a Saccheri quadrilateral is equal to its base.

23. A midsegment of a triangle is half as long as the third side.
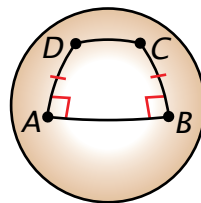
Lobachevskian Quadrilateral. ACDF is a quadrilateral in Lobachevskian geometry.

24. Copy the figure and mark the following information on it: AF ⊥ FD, CD ⊥ FD, AB = BC, FE = ED, and AF = CD.

25. What kind of quadrilateral is ACDF?

26. What kind of angles are ∠A and ∠C? Explain.

27. What kind of angles are ∠1, ∠2, ∠3, and ∠4? Explain.

28. What kind of quadrilaterals are ABEF and CBED?

29. Why is BE < AF and BE < CD?

30. State a theorem in Lobachevskian geometry about the length of the line segment that connects the midpoints of the summit and base of a Saccheri quadrilateral.
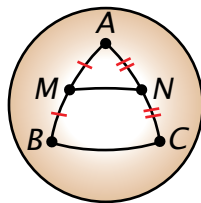
Riemannian Quadrilateral. The sphere geometry that we studied in Lesson 1 helps us understand Riemannian geometry because there are no parallel lines in either.

The figure above shows a Saccheri quadrilateral on a sphere.
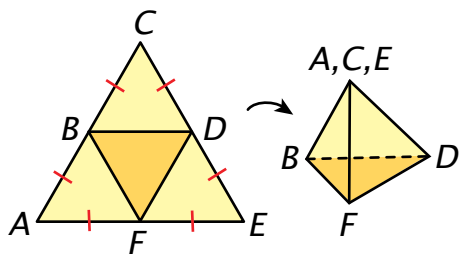
31. What kind of angles do ∠D and ∠C seem to be?

32. Does your answer agree with the table on page 703?

33. How do DC and AB seem to compare in length?

34. State a theorem in Riemannian geometry that you think corresponds to Lobachevskian Theorem 1.

Riemannian Triangle. The figure below shows a triangle on a sphere; MN is one of the triangle's midsegments.

35. State a theorem in Riemannian geometry that you think corresponds to Lobachevskian Theorem 2.

36. Do you think that a midsegment of a triangle in Riemannian geometry is parallel to the third side of the triangle? Explain.

Triangular Pyramid. In the pattern below, B, D, and F are the midpoints of the sides of equilateral △ACE. The pattern can be folded to form a triangular pyramid whose properties depend on whether it is Euclidean or non-Euclidean.
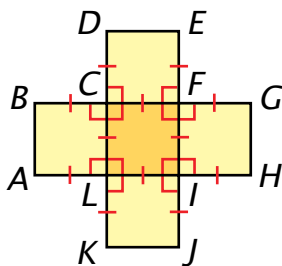
37. In Euclidean geometry, how do the edges of the pyramid compare in length with the sides of △ACE?
38. What can you conclude about the faces of the pyramid?

In Lobachevskian geometry, the pyramid formed by folding the pattern is slightly different.

39. How do the edges of its base, △BDF, compare in length with the sides of △ACE?
40. What can you conclude about its faces?

Open Box. The pattern below can be folded to form an open box in Euclidean geometry.
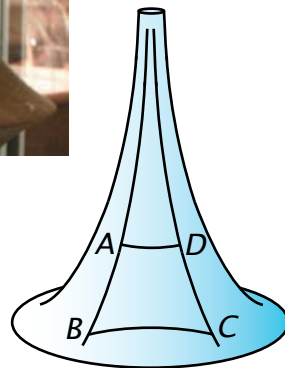
41. On the basis of the marked parts, what special shape would the box have?
42. What can you conclude about its faces?

We can show that such a pattern is impossible in Lobachevskian geometry.

43. What kind of quadrilaterals are the four faces shown in yellow?
44. How would AB, DE, GH, and JK compare in length with the sides of CFIL in Lobachevskian geometry?
45. What special type of quadrilateral is CFIL?
46. Why can't such a quadrilateral exist in Lobachevskian geometry?

Set III

Pseudosphere. The photograph below is of a model of a *pseudosphere*, a surface invented by the Italian geometer Eugenio Beltrami.*

The surface of a pseudosphere can be used to represent a part of a plane in Lobachevskian geometry. In the figure above, ABCD is a Saccheri quadrilateral.

What conclusions can you draw about the figure?

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*Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions, by Thomas F. Banchoff (Scientific American Library, 1990).

The Triangle Angle Sum Theorem Revisited

We live in a mysterious universe. Before Einstein and relativity, physical space was assumed to be Euclidean. In the nineteenth century, mathematicians realized that the geometry of Euclid is not the only one possible, but none of them would have guessed that a non-Euclidean geometry might better describe the universe. Yet Einstein's theory of relativity and the experiments confirming it suggest that a geometry other than Euclid's does.*

At about the time that Lobachevsky and Bolyai were developing the non-Euclidean geometry to which we have been referring as Lobachevskian, Karl Friedrich Gauss measured the angles of a huge triangle. You know that, in Euclidean geometry, the sum of the angles of a triangle is exactly 180°. According to the geometry of Lobachevsky, it is *less* than 180°, whereas, in Riemannian geometry, it is *more* than 180°.

Albert Einstein (1879–1955)

*For more information about the possible geometries of space, see *Einstein's Legacy*, by Julian Schwinger (Scientific American Library, 1986).

The sides of the triangle that Gauss measured were formed by light rays sent between three mountain tops in Germany. The sum of the angles of the triangle turned out to be just a few seconds more than 180°. Gauss's measurements were as accurate as was possible at the time, but Gauss understood that the fact that the sum was slightly more than 180° could have been due to experimental error. Indeed, Gauss knew that the mountain-top triangle was much too small to determine whether physical space is non-Euclidean, even though the lengths of its sides ranged from 43 to 66 miles. According to both Lobachevskian and Riemannian geometry, the smaller the triangle, the closer the sum of the measures of its angles is to 180°. If space can be described by either of these non-Euclidean geometries, then a triangle with three stars as its vertices, rather than three mountain tops, would be required to determine which model is correct.

One of the consequences of the fact that the sum of the measures of the angles of a triangle in the non-Euclidean geometries is not 180° is that scale models cannot exist in these geometries. In other words, if two figures are not the same size, they cannot have the same shape!

It is no wonder that, with such surprising results as these, it took a long time for the non-Euclidean geometries to be taken seriously. We know enough about Lobachevskian geometry to be able to prove theorems that cover these observations.

Lobachevskian Theorem 3
The sum of the angles of a triangle is less than 180°.

Corollary to Lobachevskian Theorem 3
The sum of the angles of a quadrilateral is less than 360°.

Lobachevskian Theorem 4
If two triangles are similar, they must also be congruent.

Gauss, the mathematician who measured the triangle on the mountain tops, was probably the first person in history to realize that the geometry of Euclid and the geometry of our universe are not necessarily the same. At the time of the development of Lobachevskian geometry, Gauss wrote to a friend: "I am becoming more and more convinced that the necessity of our [Euclidean] geometry cannot be proved, at least not by human reason." With regard to the fact that the theorems of the non-Euclidean geometries seem to be absurd, he wrote:

> It seems to me that we know . . . too little, or too nearly nothing at all, about the true nature of space, to consider as *absolutely impossible* that which appears to us unnatural.
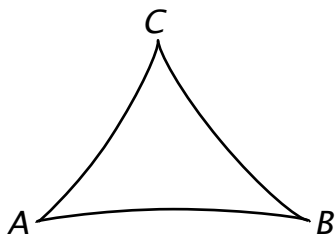
In the twentieth century, British scientist J. B. S. Haldane observed that the universe may not be only stranger than we imagine, but it may be stranger than we *can* imagine.

Exercises

Set I

Complete the proofs of the theorems of this lesson by answering the questions.
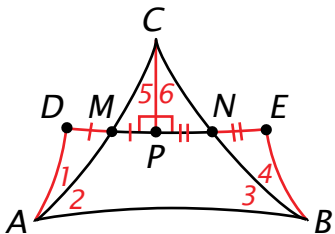
Lobachevskian Theorem 3. The sum of the angles of a triangle is less than 180°.

Given: △ABC.
Prove: ∠A + ∠B + ∠C < 180°.

Proof
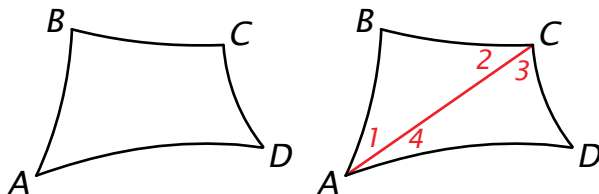
Let M and N be the midpoints of sides AC and BC, and draw MN. Through C, draw CP ⊥ MN. Choose points D and E on line MN so that MD = MP and NE = NP. Draw AD and BE.

1. △ADM ≅ △CPM and △BEN ≅ △CPN. Why?

2. Which angles of ADEB are right angles?

3. Which sides of ADEB are equal?

4. What kind of quadrilateral is ADEB?

5. ∠DAB and ∠EBA are acute. Why?

6. What can be concluded about ∠DAB + ∠EBA from the fact that ∠DAB < 90° and ∠EBA < 90°?

Because ∠DAB = ∠1 + ∠2 and ∠EBA = ∠3 + ∠4, it follows that ∠1 + ∠2 + ∠3 + ∠4 < 180°.

7. To which angles in the figure are ∠1 and ∠4 equal?

8. Why is ∠5 + ∠2 + ∠3 + ∠6 < 180°?

9. Why does it follow from this that ∠CAB + ∠CBA + ∠ACB < 180°?

Corollary to Lobachevskian Theorem 3. The sum of the angles of a quadrilateral is less than 360°.

Given: Quadrilateral ABCD.
Prove: ∠A + ∠B + ∠C + ∠D < 360°.
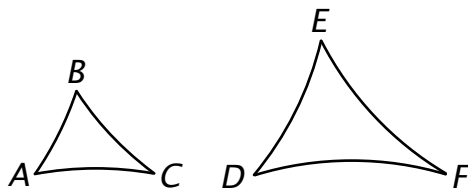
Proof
Draw AC.

10. ∠1 + ∠2 + ∠B < 180° and ∠3 + ∠4 + ∠D < 180°. Why?

11. ∠1 + ∠2 + ∠B + ∠3 + ∠4 + ∠D < 360°. Why?

12. ∠1 + ∠4 = ∠BAD and ∠2 + ∠3 = ∠BCD. Why?

13. ∠BAD + ∠B + ∠BCD + ∠D < 360°. Why?

Lobachevskian Theorem 4. If two triangles are similar, they must also be congruent.
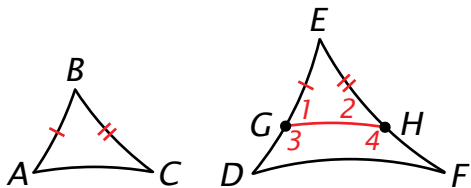
Given: △ABC ∼ △DEF.
Prove: △ABC ≅ △DEF.

Proof
By hypothesis, △ABC ∼ △DEF.

14. Why is ∠D = ∠A, ∠E = ∠B, and ∠F = ∠C?

Our proof will be indirect. Suppose that △ABC and △DEF are not congruent. If this is the case, then BA ≠ ED and BC ≠ EF, because, if either pair of these sides were equal, then the triangles would be congruent by ASA.

We will now copy the smaller triangle on the larger one as shown above. Choose point G on ED so that EG = BA and point H on EF so that EH = BC. Draw GH.

15. Why is △GEH ≅ △ABC?

It follows that ∠1 = ∠A and ∠2 = ∠C because they are corresponding parts of these triangles.

16. Why is ∠D = ∠1 and ∠F = ∠2?

17. Why is ∠1 + ∠3 = 180° and ∠2 + ∠4 = 180°?

18. Why is ∠1 + ∠2 + ∠3 + ∠4 = 360°?

19. Why is ∠D + ∠F + ∠3 + ∠4 = 360°?

20. If we assume from the figure that DGHF is a quadrilateral, the equation in Exercise 19 is impossible. Why?

Because we have arrived at a contradiction, our assumption that △ABC and △DEF are not congruent is false! So △ABC ≅ △DEF.

Set II

Lobachevskian Triangles. In Lobachevskian geometry, the sum of the angles of every triangle is less than 180°. The larger the triangle's area, the smaller the angle sum is.
  Use these facts to decide on answers to the following questions about triangles in Lobachevskian geometry.

21. If two angles of one triangle are equal to two angles of another triangle, can you
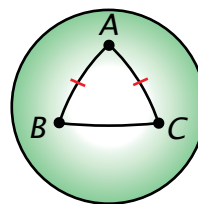
conclude that the angles of the third pair are equal? Explain.

22. What can you conclude about the acute angles of a right triangle?

23. What can you conclude about the measure of each angle of an equilateral triangle?

24. How does the measure of each angle of an equilateral triangle depend on the lengths of its sides?

Triangle on a Sphere. The figure below illustrates an isosceles △ABC with AB = AC on a sphere.
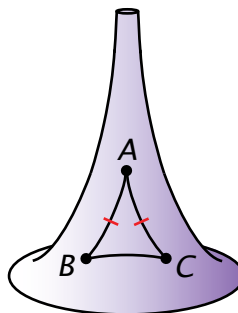
On the basis of the figure, what

25. seems to be true about ∠B and ∠C?

26. seems to be true about ∠A + ∠B + ∠C?

27. kind of geometry seems to be illustrated?

Triangle on a Pseudosphere. The figure below illustrates an isosceles △ABC with AB = AC on a pseudosphere.
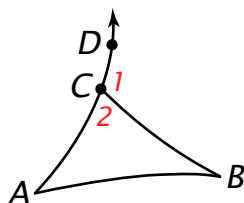
On the basis of the figure, what

28. seems to be true about ∠B and ∠C?

29. seems to be true about ∠A + ∠B + ∠C?

30. kind of geometry seems to be illustrated?
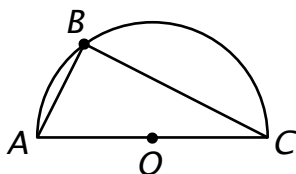
Exterior Angles. In the figure below, △ABC represents a triangle in Lobachevskian geometry and ∠1 is one of its exterior angles.

In both Euclidean geometry and Lobachevskian geometry, ∠1 > ∠A and ∠1 > ∠B.
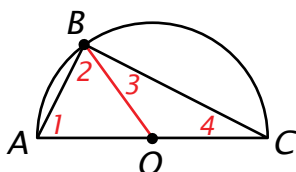
31. What else can you conclude about ∠1 in Euclidean geometry?

32. Use the figure to show whether or not your conclusion about ∠1 is also true in Lobachevskian geometry.

Angle Inscribed in a Semicircle. In the figure below, ∠ABC is inscribed in a semicircle.

33. What can you conclude about ∠ABC in Euclidean geometry?

Radius OB has been added in the figure below.
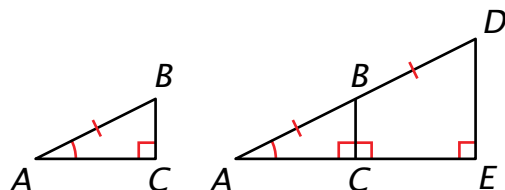
34. Use the numbered angles to show whether or not your conclusion about ∠ABC is also true in Lobachevskian geometry.

Magnification and Distortion. The following photographs are an exaggerated illustration of the fact that a figure cannot be enlarged in Lobachevskian geometry without its being distorted.
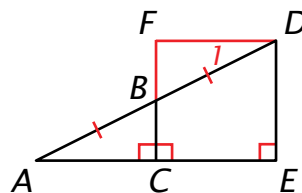
In the figure at the left below, △ABC is a right triangle with right ∠C. In the figure at the right, the triangle has been enlarged by extending line AB so that AD = 2AB and by drawing DE ⊥ line AC.

In Euclidean geometry, what can you conclude about

35. △ADE and △ABC? Why?

36. ∠D and ∠ABC?

37. the lengths of the sides of △ADE with respect to the corresponding sides of △ABC?

In the figure below, line DF has been drawn so that ∠1 = ∠A, and BC has been extended so that it intersects DF at F.

38. Copy the figure and mark this information on it.

In both Euclidean geometry and Lobachevskian geometry, what can you conclude about

39. △DBF and △ABC? Explain.

40. the lengths of BF and BC?

41. ∠F?

42. quadrilateral CFDE?

In Lobachevskian geometry, what can you conclude about

43. ∠FDE?

44. the lengths of DE and FC? Explain.

45. FC and 2BC?

46. DE and 2BC?

47. the lengths of CE and FD?

48. the lengths of CE and AC?

49. AE and 2AC?

50. How do the sides of △ADE compare in length with the corresponding sides of △ABC in Lobachevskian geometry?

Set III

Astronomical Triangle. In 1829, Lobachevsky reported that he used some astronomical measurements to find the sum of the angles of the triangle determined by Earth, the sun, and the star Sirius. He found that the sum differed from 180° by approximately 0.000000001°.

1. Does Lobachevsky's finding prove that physical space is non-Euclidean? Explain why or why not.

2. Would it be easier to prove by this method that physical space is Euclidean or that it is non-Euclidean? Explain.

Basic Ideas

Antipodal points   691
Birectangular quadrilateral   697
Great circle   690
Lobachevskian geometry   703
Riemannian geometry   703
Saccheri quadrilateral   697

Postulate

The Lobachevskian Postulate. The summit angles of a Saccheri quadrilateral are acute.   703

Theorems

87. The summit angles of a Saccheri quadrilateral are equal.   698

88. The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.   698

89. If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.   698

90. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.   698

L1. The summit of a Saccheri quadrilateral is longer than its base.   703

L2. A midsegment of a triangle is less than half as long as the third side.   703

L3. The sum of the angles of a triangle is less than 180°.   708

Corollary. The sum of the angles of a quadrilateral is less than 360°.   708
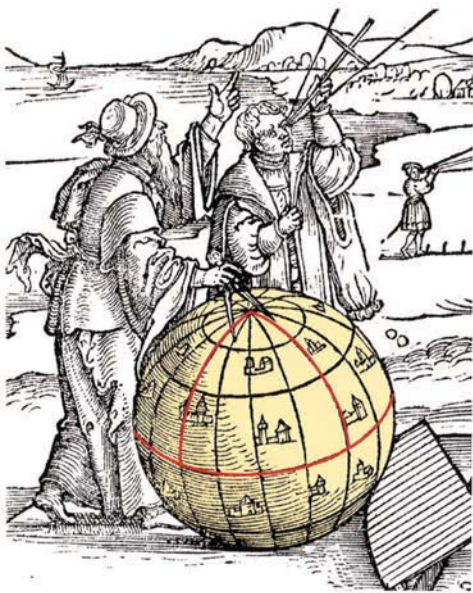
L4. If two triangles are similar, they must also be congruent.   708

Statement	*Euclid*	*Lobachevsky*	*Riemann*
Through a point not on a line, there is	exactly one line parallel to the line.	more than one line parallel to the line.	no line parallel to the line.
The summit angles of a Saccheri quadrilateral are	right.	acute.	obtuse.
The sum of the angles of a triangle is	180°.	less than 180°.	more than 180°.

Exercises

Set I

Sphere Geometry. The figure below appeared in a book published in 1533.*

Most of the curves on the globe are "lines" in sphere geometry.
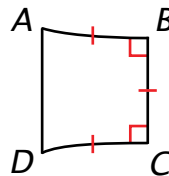
1. What is a "line" in sphere geometry?

Tell whether each of the following statements is true or false in sphere geometry.

2. Through a point not on a line, there is exactly one line perpendicular to the line.

3. In a plane, two lines perpendicular to a third line are parallel to each other.

4. If two lines intersect, they intersect in no more than one point.

5. The sum of the angles of a triangle is 180°.

6. An exterior angle of a triangle is greater than either remote interior angle.

_____
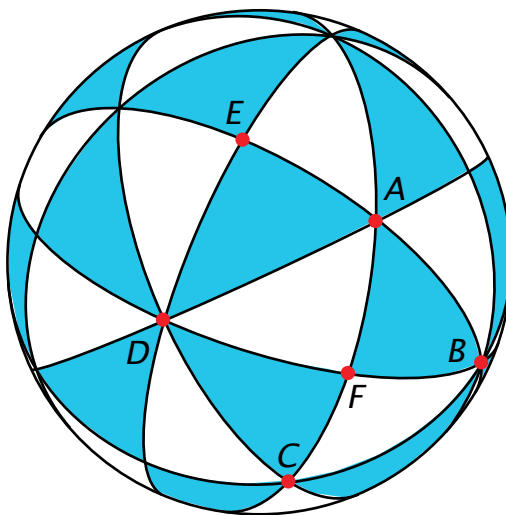
***Introduction to Geography,* by Peter Apian.

Saccheri Quadrilateral. In the figure below, ABCD is a Saccheri quadrilateral in Lobachevskian geometry; ∠B and ∠C are right angles, and AB = BC = CD.

7. Which side of ABCD is its summit?

8. What can you conclude about the length of AD?

9. Can ABCD be a rhombus? Explain.

10. What kind of angles are ∠A and ∠D?

11. Can ABCD be a rectangle? Explain.

Beach Ball. This beach ball has been divided by great circles into congruent triangles.

The angles at E and F are right angles, the angles at A and C are 60°, and the angles at B and D are 45°.

What is the sum of the measures of the angles surrounding

12. point A?

13. point B?

14. point C?
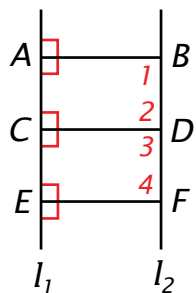
15. Make a large copy of the part of the figure bounded by quadrilateral EBCD. Label the measures of all of the angles in your figure.

16. Find the measures of the angles in quadrilateral ABCD.

17. If the opposite angles of a quadrilateral in Euclidean geometry are equal, what can you conclude?

18. Does the same conclusion appear to hold in sphere geometry? Explain.

19. What can you conclude from the fact that quadrilateral ABCD is equilateral?

20. How are the diagonals of ABCD related to each other?

21. Is this relation also true for such quadrilaterals in Euclidean geometry?

22. Find the measures of the angles in quadrilateral EAFD.

Can a quadrilateral have exactly three right angles in

23. Euclidean geometry?

24. sphere geometry?

Set II
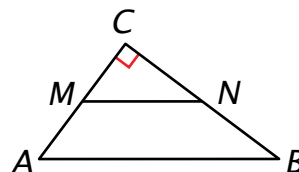
Lobachevsky's Ladder. In the figure below, l_1 and l_2 represent the two parallel side rails of a ladder. Segments AB, CD, and EF represent three rungs that are perpendicular to l_1.

In Lobachevskian geometry, no more than two of the three rungs AB, CD, and EF can have equal lengths.

25. Copy the figure and mark it to suggest that AB = CD = EF.

26. If AB = CD = EF, what kind of quadrilaterals are ABDC and CDFE?

27. In Lobachevskian geometry, what kind of angles are $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$?

28. What can you conclude about $\angle 2 + \angle 3$?

29. How does this conclusion contradict the fact that $\angle 2$ and $\angle 3$ are a linear pair?

30. What does this contradiction prove about the rungs of the ladder?

Pythagoras Meets Lobachevsky. In the figure below, $\triangle ABC$ is a right triangle and MN is one of its midsegments.
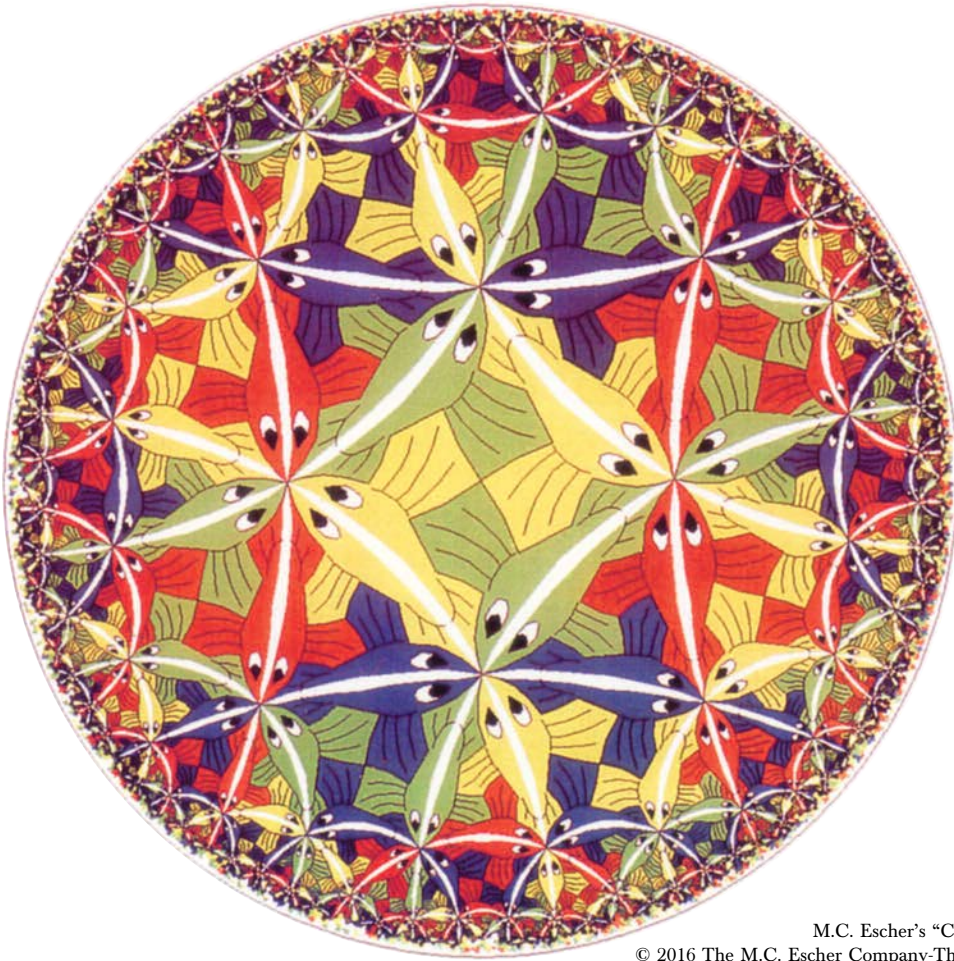
Suppose that $CA^2 + CB^2 = AB^2$ and $CM^2 + CN^2 = MN^2$.

31. On what theorem of Euclidean geometry are these equations based?

Because M and N are the midpoints of CA and CB, $CM = \frac{1}{2}CA$ and $CN = \frac{1}{2}CB$.

32. Why does it follow that $\left(\frac{1}{2}CA\right)^2 + \left(\frac{1}{2}CB\right)^2 = MN^2$?

33. Use this equation and the initial assumption that $CA^2 + CB^2 = AB^2$ to show that $MN = \frac{1}{2}AB$.

34. What theorem in Lobachevskian geometry does this result contradict?

35. What does this contradiction indicate about the Pythagorean Theorem in Lobachevskian geometry?
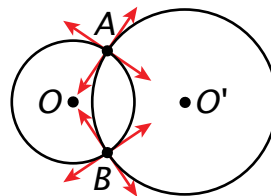
M.C. Escher's "Circle Limit III"

Lobachevsky Meets Escher. The print by Maurits Escher at the beginning of this chapter illustrates a model devised by a French mathematician, Henri Poincaré, for visualizing the theorems of Lobachevskian geometry. Silvio Levy used a computer to create the above version of it.*

To understand this model, it is necessary to know what *orthogonal circles* are.

The figure at the right shows two orthogonal circles O and O′ intersecting at points A and B. The tangents to the circles at these points have been drawn.
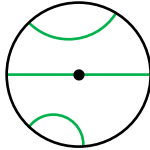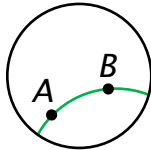
_____
***The Visual Mind: Art and Mathematics*, edited by Michele Emmer (MIT Press, 1993).

36. On the basis of this figure, define *orthogonal circles.*

In Poincaré's model of Lobachevskian geometry, points of the plane are represented by points inside a circle, and lines are represented by both the diameters of the circle and the arcs of circles orthogonal to it. Examples of some lines in this model are shown in the figure below.
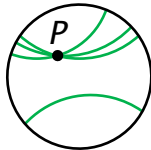
The figure below shows two points in a circle and the only arc through them that is orthogonal to the circle.

37. What postulate does this figure illustrate?
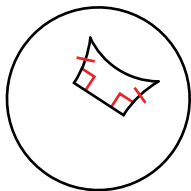
The figure below shows an "orthogonal arc" and several such arcs through point P that do not intersect it.
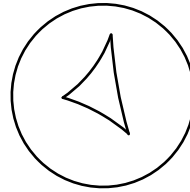
38. What idea in Lobachevskian geometry does this figure illustrate?

State a theorem or postulate in Lobachevskian geometry suggested by each of the following figures.
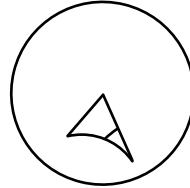
39.

40.

41.
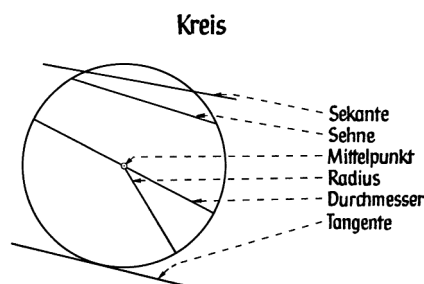
The Geometry of the Universe. According to one model of the universe, called Einstein's spherical universe, the geometrical properties of space are comparable to those of the surface of a sphere.

42. If this model is correct, what conclusion follows about the volume of the space of the universe? Explain.

43. If an astronaut were to travel far enough along a straight line, what would this model predict? Explain.

Final Review
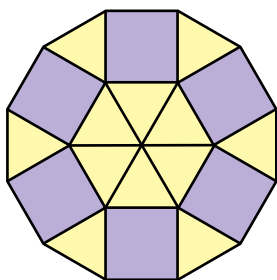
Set I

German Terms. The figure below from a German geometry book names lines and line segments related to the circle.

Kreis

- Sekante
- Sehne
- Mittelpunkt
- Radius
- Durchmesser
- Tangente

1. Which word in the figure is identical in German and English?

2. It appears from the figure that a *durchmesser* is a *sehne* of a *kreis* that contains its *mittelpunkt.* Explain.

3. What is the difference between a *sekante* and a *tangente*?

Regular Polygons. The figure below consists of equilateral triangles and squares.
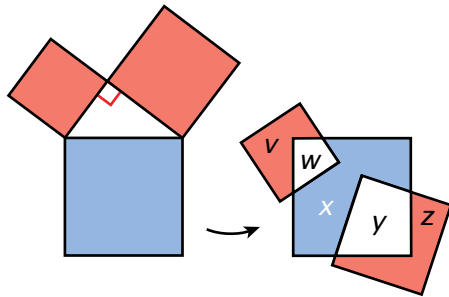
4. What other regular polygons do you see in it?

5. What properties do all regular polygons have in common?

6. What are the measures of the angles of the rhombuses in the figure that are not squares?

7. What other type of quadrilateral in the figure has angles of these measures?

8. What is the sum of the measures of the angles of a regular hexagon?

9. What is the sum of the measures of the angles of a regular dodecagon?

What Follows? Complete the statements of the following postulates and theorems.

10. The area of a parallelogram is the product of . . .

11. If a line through the center of a circle bisects a chord that is not a diameter, it . . .

12. If two points lie in a plane, the line that contains them . . .

13. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, . . .

14. Triangles with equal bases and equal altitudes . . .

15. Two nonvertical lines are parallel iff their slopes . . .

16. If a line is perpendicular to a radius at its outer endpoint, it is . . .

17. If two planes intersect, they intersect in . . .

18. The length of a diagonal of a cube with edges of length e is . . .

19. If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides . . .

20. Each leg of a right triangle is the geometric mean between . . .

21. The volume of a pyramid is one-third of the product of . . .

22. The ratio of the perimeters of two similar polygons is equal to . . .

23. Two nonvertical lines are perpendicular iff the product of their slopes . . .

24. A secant angle whose vertex is outside a circle is equal in measure to . . .

Pythagorean Squares. In the figure below, the squares on the legs of a right triangle have been moved so that they partly overlap the square on its hypotenuse. The letters represent the areas of the five regions that result.
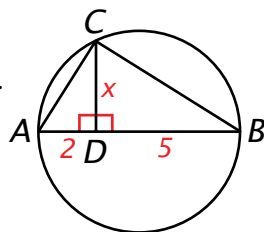
According to the Area Postulate, the area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts.

25. In terms of the letters, what is the area of the largest square?

26. Which is greater: the area shaded red, $v + z$, or the area shaded blue, x? Explain.

The Value of Pi. Brahmagupta, a seventh-century Indian mathematician, proposed the following estimate for π.
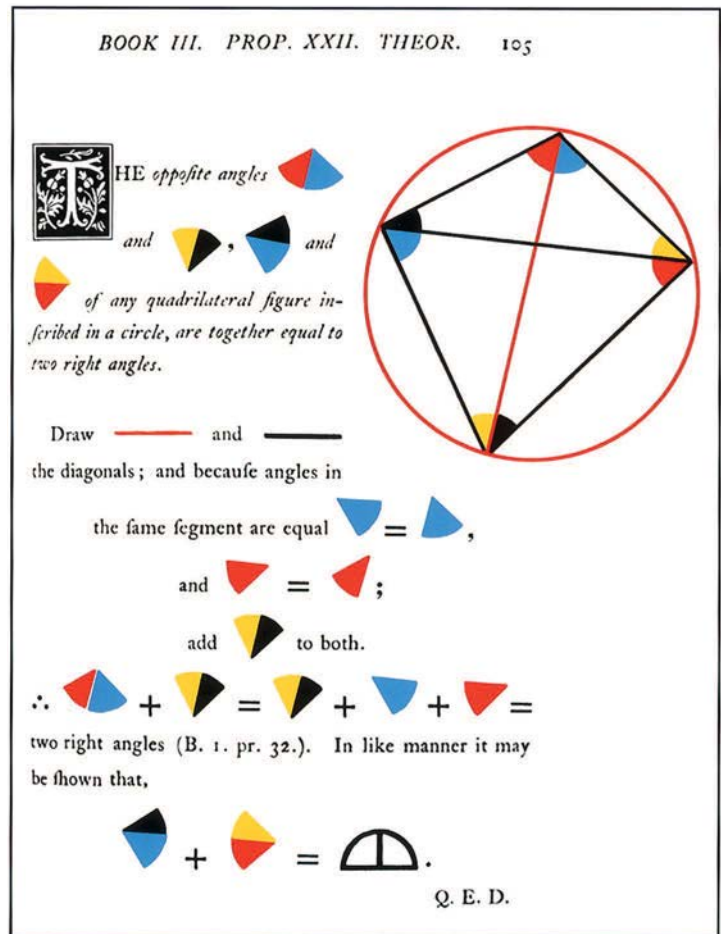
AB is a diameter of the circle and CD $\perp$ AB.

27. What can you conclude about $\triangle ABC$? Explain.

28. Write a proportion relating the lengths AD, CD, and DB.

29. Using the fact that AD = 2 and DB = 5, find CD.

Brahmagupta's estimate was CD = π.

30. Is this estimate correct? Explain.

Euclid in Color. The most colorful edition of Euclid's *Elements* was published in London in 1847. The figure above shows a page from this book.

31. What theorem is being proved?

32. Why are the angles that are labeled with the same color equal?

33. Why is the part of the proof shown below true?

34. What do you think the symbol means?

Largest Package. The rule for the largest rectangular package that you can mail at the post office is:
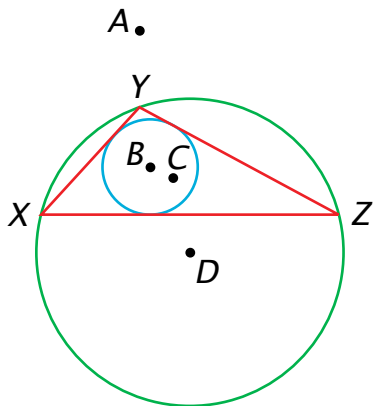
The length of its longest side plus the distance around its thickest part is less than or equal to 130 inches.

A package has dimensions 10 inches, 20 inches, and 75 inches.

35. What is its volume?

36. Would the post office accept it? Explain.

37. What is the length of the edge of the largest *cube* that you can mail?

38. What is its volume?

Four Centers. Points A, B, C, and D are the four centers of △XYZ.

What is the name of

39. point A, the point in which the lines containing the altitudes of △XYZ are concurrent?

40. point B?

41. point C, the point in which the medians of △XYZ are concurrent?

42. point D?
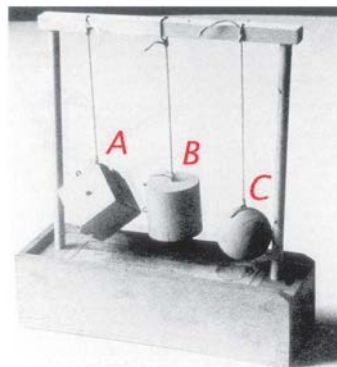
Which of these points is

43. equidistant from the sides of the triangle?

44. equidistant from its vertices?

45. the triangle's balancing point?

46. not collinear with the other three?

Formulas. What is each of the following formulas used for?

47. $c = \pi d$.

48. $d = \sqrt{l^2 + w^2 + h^2}$.

49. $A = \dfrac{1}{2} h(b_1 + b_2)$.

50. $h = \dfrac{\sqrt{3}}{2} s$.

Kindergarten Toy. The photograph from the Milton Bradley catalog of 1889 shown below is of a toy for kindergarten children. Three basic geometric solids are suspended by strings.

51. What does each solid appear to be?
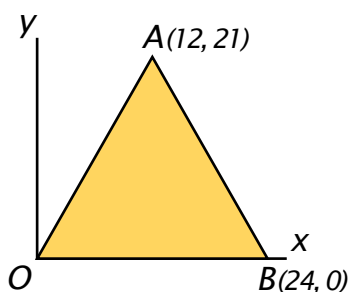
Write expressions for the volume and total surface area of

52. solid A in terms of the length, e, of each of its edges.

53. solid B in terms of its altitude, h, and the radius, r, of its bases.

54. solid C in terms of its radius, r.

Steep Roof. The steep roof of an A-frame house helps it to shed snow easily.
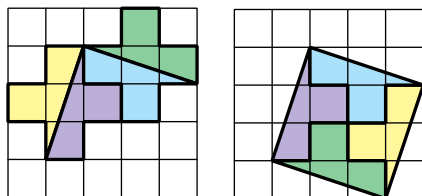
The figure below is a side view of an A-frame roof in a coordinate system; the units are in feet.

55. Find the slopes of OA and AB.

56. Find ∠AOB, the angle of inclination of the roof, to the nearest degree.

57. Find OA and AB to the nearest foot.

58. △OAB looks equilateral. Is it?

Greek Crosses. The figures below show how two congruent Greek crosses can be cut and put together to form a square.

Given that the length of each side of one of the crosses is x units, find each of the following measures in terms of x.

59. The perimeter of one of the crosses.

60. The area of one of the crosses.

61. The area of the square.

62. The length of a side of the square.

63. The perimeter of the square.
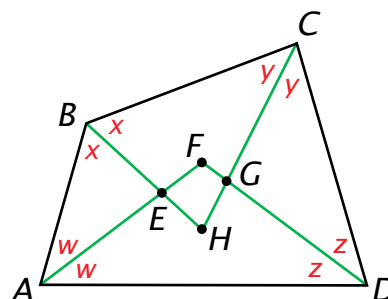
Baseball in Orbit. The smaller a planet, the weaker its gravity. If the radius of the earth were 22 miles, it would be possible to throw a baseball into orbit!*

64. Given that the orbit is circular, find the distance that the ball would travel in going once around the earth.

To be put into this orbit, the ball would have to be thrown at a speed of 100 miles per hour, something that professional pitchers can easily do.

65. Approximately how many minutes would it take the ball to return to the spot from which it was thrown?

Two Quadrilaterals. ABCD represents any convex quadrilateral. The four lines that bisect its angles intersect at E, F, G, and H to form quadrilateral EFGH.
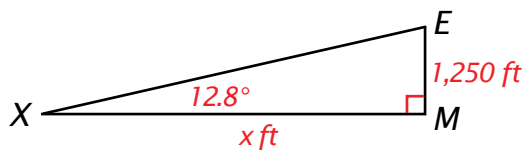
Letting the measures of the angles produced by the bisectors be w, x, y, and z as indicated, show why each of the following statements is true.

66. $w + x + y + z = 180°$.

67. $\angle F = 180° - w - z$ and $\angle H = 180° - x - y$.

68. $\angle F + \angle H = 180°$.

69. Quadrilateral EFGH is cyclic.
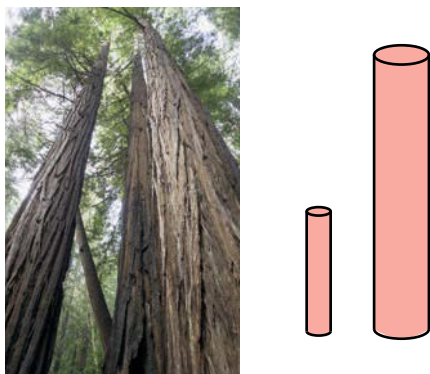
The Cosmological Milkshake, by Robert Ehrlich (Rutgers University Press, 1994).

Long Shadows. Tall buildings cast long shadows. At 3 P.M. on the winter solstice, December 21, the Empire State Building casts a shadow more than a mile long.*

70. Given that the angle of elevation of the sun at this time is 12.8° and that the Empire State Building is 1,250 feet tall, find the approximate length of its shadow.

71. How long would the shadow cast by a person 6 feet tall be at the same time?

Tree Geometry. Galileo calculated that trees cannot grow beyond a certain size without collapsing from their own weight.

The figures at the right below represent the trunks of two trees as similar cylinders.

If the ratio of their diameters is k, what is the ratio of

72. their heights?

73. the areas of their bases?

74. their volumes?

75. their weights?
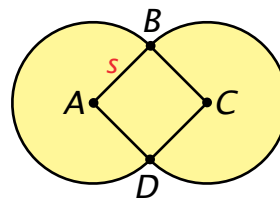
Suppose two trees are similar in shape and that one tree is 10 times as tall as the other.

76. How would corresponding cross sections of their trunks compare in area?

77. How would their trunks and branches compare in weight?

78. On the basis of these comparisons, which tree trunk would be more likely to fail because of the load that it bears?

SAT Problem. The figure at the right appeared in a problem on an SAT test.
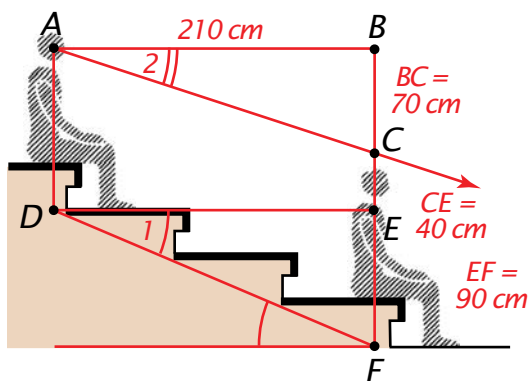
It consists of a square with side of length s and the arcs of two circles centered at A and C and having radius s.

Write an expression in terms of s for

79. the perimeter of the figure.

80. its area.

Sight Line. In the design of seating that enables each spectator to have a clear view, the sight line is projected 15 cm above the head of the spectator seated three rows below.†

Given the measurements in the figure above, find

81. ∠1, which equals the angle at which the seats rise with respect to the horizontal.

82. ∠2, the angle of the sight lines with the horizontal.

Set II

Dog Crates. Dog crates usually have the shape of rectangular solids.

Crates for beagles are 30 inches long and 22 inches high and have a volume of 12,540 cubic inches.

83. How wide are they?
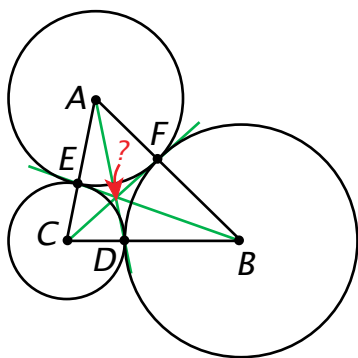
Crates for boxers are 36 inches long, 26 inches high, and 23 inches wide.

84. Are the crates for beagles and boxers similar? Explain why or why not.

Crates for Great Danes are 48 inches long.

85. Given that these crates are similar to those for beagles, find the volume of such a crate to the nearest cubic inch.
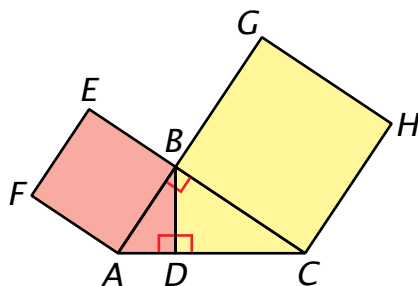
Tangent Circles. The three circles in the figure below are centered on the vertices of $\triangle ABC$ and meet on its sides at points D, E, and F.

Lines AD, BE, and CF look as if they might be concurrent.

86. Write an equation relating the lengths of the segments of the sides of $\triangle ABC$ that, if true, would show that lines AD, BE, and CF are concurrent.

87. Would this equation be true if the three circles had equal radii? Explain.

88. Is the equation true, given that the circles have radii of different lengths? Explain.

Squares on the Legs. In the figure below, squares ABEF and BCHG have been drawn on the legs of right $\triangle ABC$, and BD is the altitude to its hypotenuse.

89. $\triangle ABD \sim \triangle BCD$. Why?

90. $\dfrac{\alpha \triangle ABD}{\alpha \triangle BCD} = \left(\dfrac{AB}{BC} \right)^2$. Why?

91. How does $\dfrac{\alpha \triangle ABD}{\alpha \triangle BCD}$ compare with

$\dfrac{\alpha ABEF}{\alpha BCHG}$? Explain.

92. Is the yellow region similar to the light red one? If so, write the similarity.

Formula Confusion. On the final exam, Obtuse Ollie started confusing different formulas with the one for the area of a triangle.

For the area of a circle, he wrote $A = \dfrac{1}{2} rc$.

93. If by r and c he meant the radius and circumference of the circle, how many points out of 5 would you give Ollie? Explain.

For the volume of a sphere, Ollie wrote
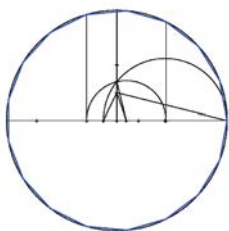
$$V = \frac{1}{2} rA.$$

94. If by r and A he meant the radius and surface area of the sphere, should Ollie get any credit for this answer? Explain.

Ollie also wrote $V = \frac{1}{2} rA$ for the volume of a right cylinder.

95. If by r and A he meant the radius and area of the lateral surface of the cylinder, should Ollie get any credit for this answer? Explain.

Regular 17-gon. The great German mathematician Carl Friedrich Gauss made a surprising discovery when he was 18 years old. He proved that a circle can be divided into 17 equal arcs with just a straightedge and compass.

96. Find the value of $N = n \sin \dfrac{180}{n}$ for a regular 17-gon to the nearest hundredth.

97. Find the value of $M = n \sin \dfrac{180}{n} \cos \dfrac{180}{n}$ for the same polygon to the nearest hundredth.
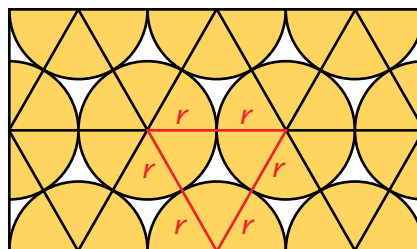
Given that the radius of a regular 17-gon is 10 centimeters, use the formula

98. $p = 2Nr$ to find its perimeter to the nearest centimeter.

99. $A = Mr^2$ to find its area to the nearest square centimeter.

100. Without calculating the numbers, how would you expect the corresponding measurements of a circle with the same radius to compare with those of the 17-gon?
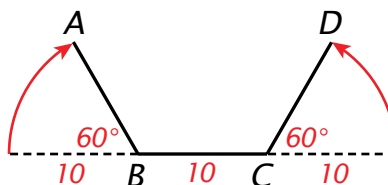
101. Check your answer to exercise 100 by calculating the circumference and area of the circle, each to the nearest square centimeter.

Packing Circles. When equal circles are packed together as closely as possible, their centers are at the vertices of equilateral triangles.

102. Find the area of one of these triangles in terms of r, the radius of the circles.

103. Find the area inside the triangle filled by the three sectors of the circles.

104. Use your answers to exercises 102 and 103 to find the percentage of the plane filled by the circles.

Rain Gutter. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending upward one-third of the sheet on each side through an angle of 60°.*
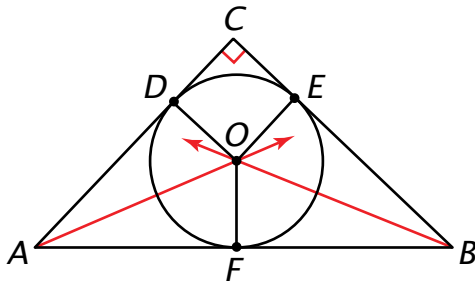
105. Copy the figure and draw perpendicular line segments from A and D to line BC to form right △AEB and right △DFC. Also draw AD.

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*From a problem in *Calculus*, by James Stewart (Brooks/Cole, 1995).

106. What can you conclude about △AEB and △DFC? Explain.

107. Why is AE ∥ DF?

108. Why is AE = DF?

109. Why is AD ∥ EF?

110. Find the length of AE.

111. Find the area of ABCD (the cross section of the water when the rain gutter is full) to the nearest square centimeter.

Incircle. In right △ABC, the bisectors of ∠A and ∠B intersect in point O.

112. Why does it follow that the bisector of ∠C also passes through point O?

113. What is point O called with respect to the triangle?
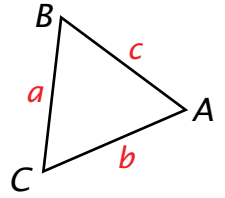
The sides of the triangle are tangent to circle O at points D, E, and F.

114. Why is CD = CE?

115. Why is OD ⊥ AC and OE ⊥ CB?

116. Why is OD ∥ EC and OE ∥ DC?

117. Why does it follow that OD = EC and OE = DC?

118. Given that CA = 20 and CB = 21, find AB.

Express each of the following lengths in terms of the given side lengths and r, the radius of the incircle.
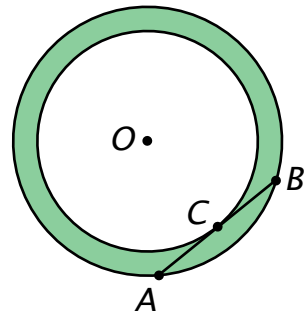
119. AD and AF.

120. BE and BF.

121. Use AB to write and solve an equation for r.

Angles and Sides. The sides opposite ∠A, ∠B, and ∠C of △ABC have lengths a, b, and c.

122. Why does it follow that $c^2 = a^2 + b^2 - 2ab \cos C$?

123. Solve this equation for cos C, given that $a = b = c$.

124. Use your calculator to find ∠C.

125. Is the result the number that you would expect? Explain.

126. Solve the equation of exercise 122 for cos C, given that $c^2 = a^2 + b^2$.

127. Use your calculator to find ∠C.

128. Is the result the number that you would expect? Explain.

Circular Track. The figure below shows a circular race track bordered by two concentric circles.

Chord AB represents one of the longest possible straight paths on the track. It is tangent to the inside of the track at C and is 200 meters long.*
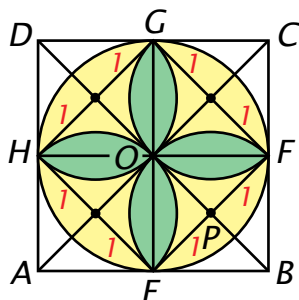
129. Copy the figure and draw OA and OC. Let R be the length of OA and r be the length of OC.

130. AB ⊥ OC. Why?

131. OC bisects AB. Why?

132. $R^2 - r^2 = 10{,}000$. Why?

133. Find the area of the track to the nearest square meter.

Mathematical Nuts, by Samuel I. Jones (S. I. Jones Co., 1932).
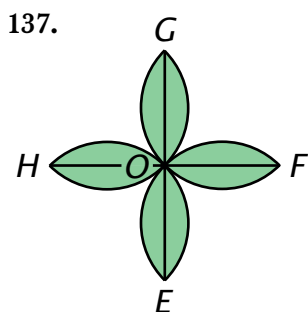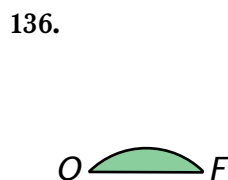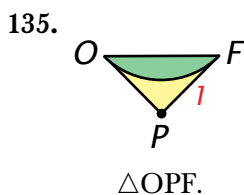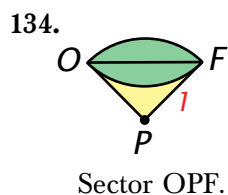
da Vinci Problem. Leonardo da Vinci dedicated a hundred pages of his notebooks to geometry. He made many drawings of figures bounded by circular arcs in which the areas of the parts were related in simple ways.*
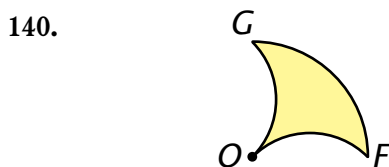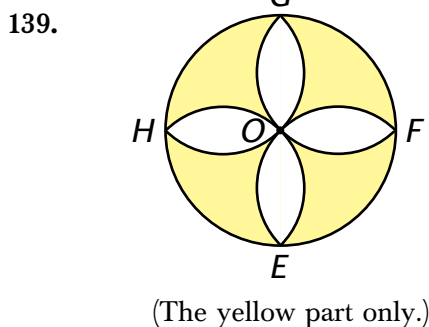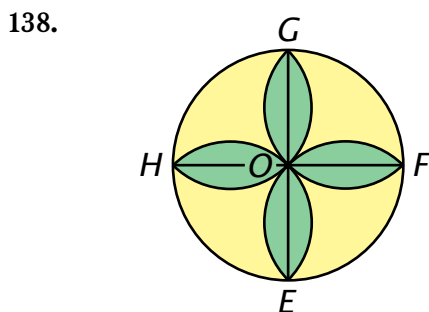
In the figure below, square ABCD is circumscribed about circle O, and square EFGH is inscribed in it. Four semicircles have been drawn with the sides of EFGH as their diameters; each diameter is 2 units long.
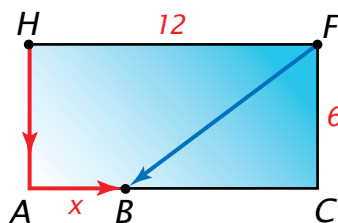
Find the area of each of the following parts of the figure.

134.

Sector OPF.

135.

△OPF.

136.

137.

138.

139.

(The yellow part only.)

140.

A Problem from Ancient India.[†] A fish at corner F of a rectangular pool sees a heron at corner H looking at him. As the fish swims along FB, the heron walks along the shore from H to A to B, arriving at the same time.
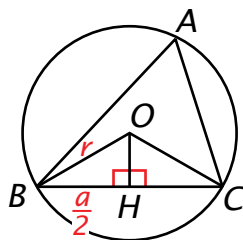
141. Given that the pool is 12 units by 6 units and that the heron and the fish move at the same speed, find the distance that the fish swam.

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*The Unknown Leonardo, edited by Ladislao Reti (McGraw-Hill, 1974).

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[†]*Geometry Civilized: History, Culture, and Technique*, by J. L. Heilbron (Clarendon Press, 1998).

Circumradius. The figure on this stamp from Guinea-Bissau can be used to derive an equation relating a triangle and its circumcircle.

The figure at the right shows △ABC with circumcircle O; OH ⊥ BC, BC = a, and OB = r.
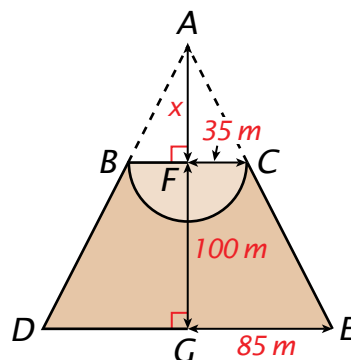
142. State the theorem that tells us that, regardless of its shape, △ABC has a circumcircle.

143. Why is BH = $\frac{a}{2}$?

144. Why is ∠BOH = $\frac{1}{2}$∠BOC?

145. Why is ∠BOC = $m\widehat{BC}$?

146. Why is ∠A = $\frac{1}{2}m\widehat{BC}$?

147. Why is ∠BOH = ∠A?

148. Why is sin ∠BOH = $\dfrac{\frac{a}{2}}{r}$?

149. Why does it follow that $\dfrac{a}{\sin A} = 2r$?

150. Given that the sides of △ABC opposite ∠B and ∠C have lengths b and c, why does it follow that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2r$?

151. Use the equation of exercise 149 to find the radius of the circumcircle of a triangle for which $a = 3$ cm and ∠A = 30°.

152. Draw a figure to show that your answer is reasonable.

Conical Mountain. The unusual mountain in Iran in the photograph below was shaped by a gaseous spring. Its crater once contained a lake.*

In the side view of the mountain below, the crater is represented as the lower part of a circular cone, and the bowl of the crater is represented as half of a sphere.

153. Use the numbers in the figure to find x, the length of AF.

Find each of the following volumes to the nearest cubic meter.

154. The volume of the solid cone whose altitude is AG.

155. The volume of the solid cone whose altitude is AF.

156. The volume of the bowl of the crater.

157. Use your answers to exercises 154 through 156 to estimate the volume of material in the actual mountain to the nearest 100,000 cubic meters.

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*Below from Above, by George Gerster (Abbeville Press, 1986).