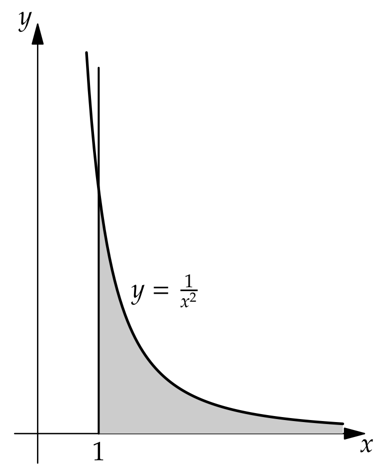
6.5 Improper integrals

As we've seen, we use the definite integral $\int_a^b f$ to compute the area of the region under the graph of $y = f(x)$ along the interval $[a, b]$. By definition, these integrals can only be used to compute areas of bounded regions. In some situations, however, we are interested in *unbounded* regions—these are regions that extend "towards infinity" in at least one direction. Yet, many unbounded regions still have finite area.

We start with a basic example of this phenomenon:

Problem 6.16: What is the area of the region bordered by the curve $y = \dfrac{1}{x^2}$, the line $x = 1$, and the x-axis?

Solution for Problem 6.16: We sketch a picture of this region at right. Notice that this region is unbounded: the region extends towards $+\infty$ as x grows large. Even though this region is unbounded, we can attempt to determine its area. We certainly can compute the area of the portion of the region to the left of $x = b$ (for any $b > 1$) as the definite integral



$$\int_1^b \frac{1}{x^2}\,dx.$$

As b grows larger, we expect that the area under the curve on $[1, b]$ approaches the area of the entire region under the curve on $[1, +\infty)$. Specifically, this area is

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx.$$

The integral is easy to evaluate:

$$\int_1^b \frac{1}{x^2}\,dx = \left. -\frac{1}{x}\right|_1^b = 1 - \frac{1}{b}.$$

Thus, when we take the limit, we get that the area of the region is

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right) = 1 - 0 = 1.$$

□

Note the "paradox" here: even though the region is unbounded, it has finite area. Problem 6.16 suggests a logical definition:

Definition: Let f be a continuous function and $a \in \mathbb{R}$ such that $(a, \infty) \subseteq \mathrm{Dom}(f)$. We define the **improper integral**

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

provided the limit is defined. If the limit is defined and is not $\pm\infty$, we say that the improper integral **converges**. Otherwise, we say that the improper integral **diverges**.

There's an obviously similar definition for improper integrals in the other direction:

Definition: Let f be a continuous function and $b \in \mathbb{R}$ such that $(-\infty, b) \subseteq \text{Dom}(f)$. We define the **improper integral**

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx,$$

provided the limit is defined. If the limit is defined and is not $\pm\infty$, we say that the improper integral **converges**. Otherwise, we say that the improper integral **diverges**.

Let's generalize Problem 6.16:

Problem 6.17: Let r be a real number. Compute

$$\int_{1}^{\infty} \frac{1}{x^r}\,dx.$$

Solution for Problem 6.17: By definition, we compute the improper integral by writing a limit. If $r \neq 1$, then we have:

$$\lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^r}\,dx = \lim_{b\to\infty} -\frac{1}{(r-1)x^{r-1}}\bigg|_{1}^{b}.$$

This equals

$$\lim_{b\to\infty} \frac{1}{r-1}\left(1 - \frac{1}{b^{r-1}}\right).$$

If $r > 1$, then the term $\frac{1}{b^{r-1}}$ approaches 0 as b approaches ∞. Thus, in this case, the improper integral converges to $\frac{1}{r-1}$.

If $r < 1$, then the term $\frac{1}{b^{r-1}}$ grows without bound as b approaches ∞. Thus, the integral diverges. We might also write

$$\int_{1}^{\infty} \frac{1}{x^r}\,dx = \infty \qquad \text{if } r < 1.$$

Our original integration was not valid for $r = 1$, so we have to do that case separately:

$$\lim_{b\to\infty} \int_{1}^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} (\log x)\bigg|_{1}^{b} = \lim_{b\to\infty} (\log b).$$

As b goes towards infinity, this grows without bound, so the integral diverges.

In summary:

$$\int_{1}^{\infty} \frac{1}{x^r}\,dx = \begin{cases} \frac{1}{r-1} & \text{if } r > 1, \\ \text{diverges} & \text{if } r \leq 1. \end{cases}$$

□

The next problem is another common example of an improper integral:

Problem 6.18: Compute $\displaystyle\int_{0}^{\infty} e^{ax}\,dx$, where a is a real number.

Solution for Problem 6.18: We compute, for $a \neq 0$ (we'll investigate $a = 0$ at the end):

$$\lim_{b\to\infty} \int_{0}^{b} e^{ax}\,dx = \lim_{b\to\infty} \frac{1}{a}(e^{ab} - 1) = \frac{1}{a} \lim_{b\to\infty} (e^{ab} - 1).$$

If a is positive, then $\lim\limits_{b\to\infty} e^{ab} = \infty$, so the integral diverges. If a is negative, then $\lim\limits_{b\to\infty} e^{ab} = 0$, so the integral equals $-\frac{1}{a}$. (Note this is a positive number when a is negative, so this answer makes sense.) Finally, if $a = 0$, then the integral is $\int_0^\infty 1\,dx$, which clearly diverges.

Thus, the integral diverges for nonnegative exponents, and converges for negative exponents. □

The result of Problem 6.18 is typically written as follows: if $r > 0$, then

$$\int_0^\infty e^{-rx}\,dx = \frac{1}{r}.$$

Problem 6.19: Suppose f and g are continuous functions on $[a, \infty)$ and $f(x) \le g(x)$ for all $x \ge a$.

(a) Show that, if $\int_a^\infty f$ and $\int_a^\infty g$ both converge, then

$$\int_a^\infty f \le \int_a^\infty g.$$

(b) Show that if both functions are positive, and $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges.

(c) Show that if both functions are positive, and $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

Solution for Problem 6.19:

(a) For any $b \ge a$, we have $(g - f)(x) \ge 0$ for all $x \in [a, b]$, thus

$$\int_a^b (g - f)(x)\,dx \ge 0.$$

Therefore,

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx,$$

and since limits preserve non-strict inequalities, we conclude that

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \le \lim_{b\to\infty} \int_a^b g(x)\,dx = \int_a^\infty g(x)\,dx.$$

(b) Define a function

$$F(x) = \int_a^x f(t)\,dt.$$

Note that F is an increasing function (since $f(x) \ge 0$ for all $x \ge a$), and that $\int_a^\infty f(x)\,dx = \lim\limits_{x\to\infty} F(x)$, if this limit exists. Also, since $0 \le f(x) \le g(x)$ for all $x \ge a$, we have

$$0 \le F(x) = \int_a^x f(t)\,dt \le \int_a^x g(t)\,dt \le \int_a^\infty g(t)\,dt.$$

Thus F is increasing and has an upper bound (namely, $\int_a^\infty g(t)\,dt$, which by assumption converges), so by the result of Problem 6.5, the limit

$$\lim_{x\to\infty} F(x) = \int_a^\infty f(t)\,dt$$

exists, so the integral converges.

(c) This is just the contrapositive statement to part (b), so there is nothing additional to prove.

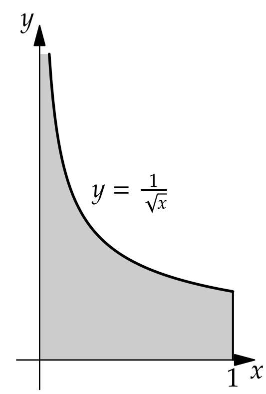
□

WARNING!! We can only use the comparison tests in parts (b) and (c) of Problem 6.19 if both functions are positive. As a trivial example, if $f(x) = -1$ and $g(x) = 0$, then for any $a \in \mathbb{R}$, $\int_a^\infty g = 0$, so $\int_a^\infty g$ converges, but $\int_a^\infty f$ diverges.

Thus far in this section, we have looked at improper integrals that compute areas of regions that are unbounded in the x-direction. There is another type of improper integral that occurs when the region that we are examining is unbounded in the y-direction, as in the following example:

Problem 6.20: Compute $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution for Problem 6.20: Sketching the graph will immediately show the issue. We have $\displaystyle\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$. So the area under $y = \dfrac{1}{\sqrt{x}}$ is potentially infinite (and in fact the function is not even defined at 0).



We can do essentially the same thing we did for improper integrals with a limit of integration of $\pm\infty$. We define

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx.$$

Note the "0^+"—since we only care about the interval (0,1], we only care about what happens to the right of 0.

This integral is now easy to compute:

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_c^1 = 2 - 2\sqrt{c}.$$

As $c \to 0^+$, this approaches 2. Hence

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

Once again, a seemingly infinite area turns out to be finite. □

We can generalize the definition from Problem 6.20:

Definition: Suppose f is a function, continuous on $(a, b]$, such that $\displaystyle\lim_{x\to a^+} f(x) = \pm\infty$. We define the **improper integral**

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx,$$

provided this limit is defined. If the limit is defined, we say that this improper integral **converges**, and if it is undefined, we say that the improper integral **diverges**.

Of course, we can do the same thing if the function has a limit of $\pm\infty$ at the "b" end of $[a, b)$. (We will omit writing out the formal definition.)

Sidenote: Note that the above definition is consistent with our usual (non-improper) integrals. In particular, if $\int_a^b f$ is defined, then by the Fundamental Theorem of Calculus, the function

$$g(x) = \int_x^b f(t)\,dt$$

is differentiable, hence continuous, and thus

$$\int_a^b f(t)\,dt = g(a) = \lim_{x\to a^+} g(x) = \lim_{x\to a^+} \int_x^b f(t)\,dt.$$

We know that for regular (not improper) integrals, we can break them apart at any point into two separate integrals. Specifically, if $c \in (a, b)$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

This is also how we evaluate integrals that are improper at both ends, as in the following example:

Problem 6.21: Compute $\displaystyle\int_0^\infty \frac{1}{x^r}\,dx$ for all $r > 0$ (or determine when it diverges).

Solution for Problem 6.21: The correct thing to do with an integral that is improper at both ends is to split it somewhere in the middle. For example, we can write

$$\int_0^\infty \frac{1}{x^r}\,dx = \int_0^1 \frac{1}{x^r}\,dx + \int_1^\infty \frac{1}{x^r}\,dx.$$

(We didn't have to pick $x = 1$ as the point at which to split them, but it seems convenient since x^r is nicely behaved at $x = 1$.) We already know by Problem 6.17 that $\int_1^\infty \frac{1}{x^r}\,dx$ converges if and only if $r > 1$. The other integral is

$$\int_0^1 \frac{1}{x^r}\,dx = \lim_{a\to 0+} \int_a^1 \frac{1}{x^r}\,dx = \lim_{a\to 0+}\left(-\frac{1}{(r-1)x^{r-1}}\bigg|_a^1 \right) = \frac{1}{r-1} \lim_{a\to 0+}\left(\frac{1}{a^{r-1}} - 1\right).$$

If $r > 1$, then the fraction gets arbitrarily large, so the limit is infinite. Thus $\int_0^1 \frac{1}{x^r}\,dx$ diverges for $r > 1$.

Hence our original doubly-improper integral is never convergent: the integral on $(0, 1]$ diverges for $r > 1$, and the integral on $[1, \infty)$ diverges for $r \le 1$. □

Important: If $(a, b) \subseteq \mathrm{Dom}(f)$ and $\displaystyle\int_a^b f(t)\,dt$ is improper at both ends of (a, b), then

$$\int_a^b f(t)\,dt = \lim_{x\to a^+} \int_x^c f(t)\,dt + \lim_{x\to b^-} \int_c^x f(t)\,dt,$$

for any $c \in (a, b)$.

As noted in the solution to Problem 6.21, it doesn't matter at which point we break up the doubly-improper integral.

Concept: We can break an integral apart as
$$\int_a^b f = \int_a^c f + \int_c^b f$$
at any $c \in (a, b)$ that we choose. Thus, choose c to be as convenient as possible.

We will leave it as an exercise to prove this. Also, it is not correct to try to take a shortcut and deal with both ends of a double-improper integral at once. In particular:

WARNING!! $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is not the same as $\displaystyle\lim_{a\to\infty} \int_{-a}^{a} f(x)\,dx$.

The correct way to evaluate an integral over all of $\mathbb{R}$ is to choose $c \in \mathbb{R}$, and then compute
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{c} f(x)\,dx + \lim_{b\to+\infty} \int_{c}^{b} f(x)\,dx.$$
We will leave it as an exercise to explore this further.

We also have to be a bit cautious when dealing with functions with domains that are not all of $\mathbb{R}$. Integrals of such functions might be improper but not immediately appear so. For example:

Problem 6.22: Compute $\displaystyle\int_{-2}^{3} \frac{1}{x^2}\,dx$.

Solution for Problem 6.22: If you weren't paying close attention, you might do this:

Bogus Solution:
$$\int_{-2}^{3} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{-2}^{3} = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.$$

We can't do this, because the function is not defined at 0! To be a little more precise, the function $\frac{1}{x^2}$ does not have an antiderivative on the interval $[-2, 3]$, because it is not defined at $x = 0$, so we cannot apply the Fundamental Theorem of Calculus.

In order to evaluate the integral, we need to break it up into a sum of two improper integrals at the point at which the function is undefined:
$$\int_{-2}^{3} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{3} \frac{1}{x^2}\,dx.$$
As we saw in Problem 6.21, both of these diverge. Thus, the original integral itself diverges. □

More generally, when computing something like $\displaystyle\int_{-1}^{1} \frac{dx}{x}$, it might be tempting to say "$\frac{1}{x}$ is an odd function, so the integral from -1 to 0 will cancel out the integral from 0 to 1, and thus the overall integral is 0." This is also the result that naive calculation will give:

Bogus Solution:
$$\int_{-1}^{1} \frac{dx}{x} = \log|x|\bigg|_{-1}^{1} = \log(1) - \log(1) = 0.$$

But this is not correct! The only way legally to evaluate this integral is to break it up into its improper parts.

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}.$$

Neither part converges, so the integral diverges.

Exercises

6.5.1 Compute the following improper integrals:

(a) $\displaystyle\int_3^\infty \frac{1}{(2x-1)^2}\,dx$ (b) $\displaystyle\int_2^\infty \frac{1}{x(\log x)^2}\,dx$ (c) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ (d) $\displaystyle\int_0^2 \frac{1}{4-x^2}\,dx$

6.5.2

(a) Compute $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$.

(b) Compute $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

6.5.3 Compute $\displaystyle\int_0^\infty x^2e^{-x}\,dx$.

6.5.4 Show that it doesn't matter at which point we break up a doubly-improper integral. Specifically, show that, for any $c, d \in (a, b)$, if $\int_a^c f$ and $\int_c^b f$ converge, then $\int_a^d f$ and $\int_d^b f$ also converge, and

$$\int_a^c f + \int_c^b f = \int_a^d f + \int_d^b f.$$

Hints: 230, 97

6.5.5★

(a) Show that if $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to\infty}\int_{-a}^a f(x)\,dx$. **Hints:** 165

(b) Show that the converse of part (a) is not true; that is, it is possible that $\displaystyle\lim_{a\to\infty}\int_{-a}^a f(x)\,dx$ converges but that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ diverges. **Hints:** 82, 225

Review Problems

6.23 Compute the following:

(a) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{\log x}$ (b) $\displaystyle\lim_{x\to 0}\frac{\cos^2 x - 1}{x^2}$ (c) $\displaystyle\lim_{x\to 0}\frac{10x^2-\frac{1}{2}x^3}{e^{4x^2}-1}$ *(Source: Rice)*

6.24 Suppose a and b are nonzero real numbers. Find $\displaystyle\lim_{t\to 0}\frac{\sin at}{\sin bt}$ and $\displaystyle\lim_{t\to 0}\frac{\tan at}{\tan bt}$. **Hints:** 32

6.25 Compute

(a) $\displaystyle\int_1^\infty e^{-2x}\,dx$ (b) $\displaystyle\int_0^2 \frac{1}{x^3}\,dx$ (c) $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx$ **Hints:** 72