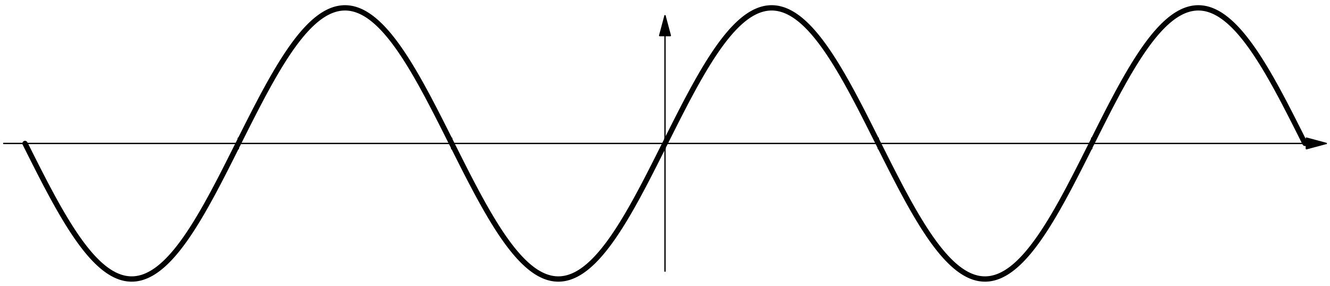

Preface

Introduction

Calculus is a branch of mathematics that, broadly speaking, introduces concepts and tools to describe and analyze functions. Although some parts of calculus were known to the ancient Greeks, Egyptians, and Chinese, the modern version of calculus that we use today was largely developed in the 17^{th} century, independently by the great mathematicians Isaac Newton and Gottfried Leibniz. Calculus is not only an important branch of mathematics in its own right, but also provides the rigorous mathematical foundation of physics, engineering, and many other branches of science.

Unfortunately, most students first learn calculus as a bag of tricks: a number of seemingly unrelated algorithms to be memorized and then endlessly applied to problem after problem without motivation. Calculus is sometimes seen as the pinnacle of high-school mathematics, and passing a college placement exam in calculus is the ultimate goal. Students who learn calculus this way—and this describes most high-school and college students who take calculus—will likely never appreciate the beauty or richness of the subject.

Our goal for this book is to present calculus with a substantial theoretical underpinning. Calculus, at its heart, is a few fundamental ideas that come together to create a rich subject, and is not a collection of definitions, formulas, and algorithms. Our hope is that students who complete this book will *understand* calculus, both as a theoretical subject and as a problem-solving tool.

Who should study calculus using this book

The target audience for this book is motivated high school students who have mastered the high school curriculum and have developed the mathematical maturity necessary to handle the level of mathematical rigor in this text. At a minimum, students must have mastered algebra, plane geometry, and trigonometry before considering continuing on to calculus. This is true for any course of calculus study, and especially important for the more rigorous calculus treatment in this book. We strongly recommend a robust precalculus curriculum, such as that in [**Ru**], before proceeding with this text. (Note: bold letters in brackets such as [**Ru**] refer to the References on page 303.)

Students—even highly skilled students—who rush into calculus too soon are likely to be frustrated and will not be able to appreciate the richness and subtleties of the subject. Such a student will only learn calculus as a set of algorithms to be memorized, and even though this may be sufficient to progress on to the next subject, it will rob the student of a key step in his or her mathematical development.

We strongly recommend that students, especially younger students, be exposed to both nontrivial problem solving and to discrete mathematics (such as combinatorics and number theory) *before* "continuing on" to calculus. Art of Problem Solving has other textbooks and online courses in both of these areas.

This book in particular is designed for students who want a deeper understanding of calculus than a mainstream high-school or college calculus text provides, and who also want exposure to a variety of non-routine calculus problems. Specifically, this book differs from a "mainstream" calculus book in two major ways:

1. A more rigorous presentation, including proofs where applicable. For example, this book begins with set theory and the construction of the real numbers, including a discussion of suprema, infima, and completeness. We cover the rigorous δ-ϵ definition of limit, which is omitted from many calculus texts. We also prove many of the important results of calculus, including the Mean Value Theorem and the Fundamental Theorem of Calculus; these results are often merely asserted without proof in standard calculus treatments.

2. An assortment of nontrivial problems. This book has fewer routine "drill-and-kill" exercises than most calculus texts, and instead has a wider array of problems that require the students to go beyond rote memorization of algorithms and instead to think more deeply about how the different aspects of calculus are interrelated. We have taken many nontrivial problems from two of the premier math contests that include calculus: the high-school level **Harvard-MIT Mathematics Tournament** (HMMT) and the college-level **William Lowell Putnam Mathematical Competition**. The HMMT is an annual math tournament for high school students, held at MIT and at Harvard in alternating years. It is run exclusively by MIT and Harvard students, most of whom themselves participated in math contests in high school. More information is available at `web.mit.edu/hmmt`. The Putnam Competition is a long-running North American undergraduate math competition held every December—2014 will be the 75$^{\text{th}}$ annual contest—which consists of *extremely* difficult problems across the entire undergraduate mathematics curriculum. More information is at `math.scu.edu/putnam`.

Students who are preparing for college calculus placement examinations may wish to supplement their study with a test-preparation workbook. The key to success on such placement examinations is repetition of routine calculations, which this textbook largely eschews in favor of a variety of more difficult, non-routine problems.

Structure of the book

Chapter 1 is a review of important foundational material: sets, real numbers, functions, graphs, trigonometry, exponentials, and logarithms. We urge students not to skip this chapter, even if they think that they already know this material. We present the material at a level of detail and rigor that students may not be used to, and will be introducing terminology and notation that is not typically covered in a precalculus class.

Chapter 2 introduces the first core idea of calculus: the limit. We present the full δ-ϵ treatment of limits, which is often not covered in a high-school calculus course, but we feel that exposure to the rigorous definition of limit is an important part of students' mathematical development. In particular, stating key definitions rigorously is necessary to prove later results, and our goal will be to prove (and not merely assert) as much as possible.

Chapters 3–5 are the heart of the subject: differentiation (Chapters 3 and 4) and integration (Chapter 5). These are the core concepts of calculus.

Chapters 6 and 7 deal with the concept of "infinity," of which most students have a vague understanding but which we will attempt to make precise. Chapter 7 also includes the very important topic of Taylor series, which is a fundamental topic of analysis.

Chapters 8 and 9 are essentially independent, and each serve as an introduction to the broader world of analysis beyond calculus. Chapter 8 covers plane curves, including curves in polar coordinates, and is a nice introduction to topics that will be covered more thoroughly in multivariable calculus (typically the next calculus course after a student has mastered the topics in this book). Finally, Chapter 9 is an introduction to the theory of differential equations, a very broad subject which we only touch on in this book, but which is the subject of (several) courses of study beyond calculus.

Throughout the book, you will see various shaded boxes and icons.

Concept: This will be a general problem-solving technique or strategy. These are the "keys" to becoming a better problem solver!

Important: This will be something important that should be learned. It might be a formula, a solution technique, or a caution.

WARNING!! Beware if you see this box! This will point out a common mistake or pitfall.

Sidenote: This box will contain material which, although interesting, is not part of the main material of the text. It's OK to skip over these boxes, but if students read them, they might learn something interesting!

Bogus Solution: Just like the impossible cube shown to the left, there's something wrong with any "solution" that appears in this box.

Most sections end with several **Exercises**. These will test students' understanding of the material that was covered in the section. Students should try to solve *all* of the exercises. Exercises marked with a $\star$ are more difficult.

All chapters conclude with several **Review Problems**. These are problems that test basic understanding of the material covered in the chapter. Students should be able to solve most or all of the Review Problems for every chapter—if unable to do so, then the student hasn't yet mastered the material, and should probably go back and read the chapter again.

All chapters also contain several **Challenge Problems**. These problems are generally more difficult than the other problems in the book, and will really test students' mastery of the material. Some of them are very, very hard—the hardest ones are marked with a $\star$. Students should not expect to be able to solve all of the Challenge Problems on their first try—these are difficult problems even for experienced problem solvers.

Many chapters will have one or more advanced sections after the end-of-chapter problems. These sections are denoted with a letter (such as 1.A). These sections are optional and often cover topics at a more theoretical level than in the main text. Eager students who work through these sections should find them rewarding, but it is acceptable to skip them.

Hints

Many problems come with one or more hints. Readers can look up the hints in the Hints section in the back of the book. The hints are numbered in random order, so that when looking up a hint to a problem students will not accidentally glance at the hint to the next problem at the same time.

It is very important that students first try to solve each problem without resorting to the hints. Only after one has seriously thought about a problem and is stuck should one seek a hint. Also, for problems which have multiple hints, use the hints one at a time; don't go to the second hint until having thought about the first one.

Solutions

The solutions to all of the Exercises, Review Problems, and Challenge Problems are in the separate Solutions Manual. There are some very important things to keep in mind:

1. Students should make a serious attempt to solve each problem before looking at the solution. Don't use the solutions book as a crutch to avoid really thinking about the problem first. Think *hard* about a problem before deciding to look at the solution. On the other hand, after serious effort has been made on a problem, students should not feel bad about looking at the solution if they are really stuck.
2. After solving a problem, it's usually a good idea to read the solution. The solutions book might show a quicker or more concise way to solve the problem, or it might have a completely different solution method.
3. If the reader is unable to solve a particular problem and has to look at the solution in order to solve that problem, he or she should make a note of it. Then, the student should come back to that problem in a week or two to make sure that he or she is able to solve it without resorting to the solution.

Resources

Here are some other good resources for students to pursue further their study of mathematics:

- The Art of Problem Solving's *Precalculus* textbook by Richard Rusczyk, in particular the chapters covering trigonometry. The other major subjects covered in *Precalculus*—complex numbers and linear algebra—are not necessary for this calculus book, but are very important for students' future math studies.
- *The Art of Problem Solving* books, by Sandor Lehoczky and Richard Rusczyk. Whereas the book that you're reading right now will go into great detail of one specific subject area—calculus—*the Art of Problem Solving* books cover a wide range of precalculus problem solving topics across many different areas of mathematics.
- The `www.artofproblemsolving.com` website. The publishers of this book also maintain the Art of Problem Solving website, which contains many resources for students:
  - a discussion forum
  - online classes
  - resource lists of books, contests, and other websites
  - a LaTeX tutorial

  - a math and problem solving Wiki
  - and much more!

- Students can hone their problem solving skills (and perhaps win prizes!) by participating in various math contests. For U.S. high school students, some of the best-known contests are the AMC/AIME/USAMO series of contests (which are used to choose the U.S. team for the International Mathematical Olympiad), the American Regions Math League (ARML), the Mandelbrot Competition, the Harvard-MIT Mathematics Tournament, and the USA Mathematical Talent Search. Links to these and many other contests are available on the Art of Problem Solving website.

Technology

Most students who study calculus will do so with the aid of a graphing calculator, and we encourage students using this book to do so as well. Once students have mastered the basics, a graphing calculator can remove some of the tedium from long calculations, and can also serve as a valuable check of students' work. Additionally, much of calculus is visual in nature, and being able to sketch, quickly and accurately, a graph of a function with a few keystrokes is very beneficial. However, students should be aware of the following cautions:

1. There is a famous saying: "garbage in, garbage out." That is, a graphing calculator is only as good as its user—if you enter bogus data into it, you will get bogus results. You also need to know how to properly use your calculator, and to make sure that it is in the correct mode (for example, while doing calculus, your calculator should be in "radians" mode and not in "degrees" mode).
2. Make sure your calculator is sufficiently sophisticated. A "scientific calculator" may not have enough features to be broadly useful for calculus. Ideally, your calculator should be able to (a) graph functions with an arbitrary viewing window, (b) solve equations numerically, and (c) numerically compute derivatives and definite integrals. (Don't worry if you don't know what all these things mean yet—that's what this book is for!) Top-of-the-line calculators are also able to do (b) and (c) symbolically (that is, in terms of variables) as well as numerically.
3. If you are planning to take the standardized calculus examination, then make sure your calculator doesn't do too much. While most calculus examinations permit (or even require) the use of calculators, a "calculator" that is actually a handheld computer or PDA will likely not be permitted. Check with the organization administering your placement test to see if they have a list of approved calculators.

We also recommend the use of symbolic computation websites. One of the best is Wolfram|Alpha (available at `wolframalpha.com`), which makes many of the features of the computational software *Mathematica* available on the web. (In fact, the author of this book used Wolfram|Alpha to check many of the calculations.)

Webpage

The Art of Problem Solving website has a page containing links to websites with content relating to material in this book, as well as an errata list for the book. This page can be found at:

`http://www.artofproblemsolving.com/BookLinks/Calculus/links.php`

If you find an error in this book, please check the above website to see if we have already posted a correction. If not, we would greatly appreciate it if you would contact us at `books@artofproblemsolving.com` and tell us about the error, so that we can issue a correction and update the errata list on the website.

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This book was written using the LaTeX document processing system. The diagrams were prepared using Asymptote.