8.1.5 For each of the following functions, describe the corresponding transformation that maps z to $f(z)$.

(a) $f(z) = z - 5 + 2i$.

(b) $f(z) = 3z - 1$.

(c) $f(z) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) z$.

(d) $f(z) = (-1 + i)z$.

(e) $f(z) = -iz - 1 + 3i$.

8.1.6 Suppose that U is on $\overline{WZ}$ in the complex plane such that $UW/UZ = a/b$. Show that if u, w, and z are complex numbers corresponding to U, W, and Z, then $u = (az + bw)/(a + b)$.

8.1.7★ Let z_1 be a complex number. Let z_2 be the complex number obtained when z_1 is rotated counterclockwise about the origin by $\frac{\pi}{2}$, and let z_3 be the complex number obtained when z_2 is rotated counterclockwise about $4 + 3i$ by $\frac{\pi}{2}$. Then z_3 is the same as the point obtained when z_1 is rotated counterclockwise about w by θ. Find w and θ.

8.2 Parallel and Perpendicular Lines

Problems

Problem 8.9: Let W and Z be points in the complex plane corresponding to nonzero complex numbers w and z, respectively, and let O be the origin.

(a) Show that if $\overleftrightarrow{WZ}$ passes through O, then z/w is a real number.

(b) Show that if z/w is a real number, then $\overleftrightarrow{WZ}$ must pass through O.

Problem 8.10: Suppose S, T, and U are points in the complex plane corresponding to distinct complex numbers s, t, and u, respectively. Three points are said to be **collinear** if a line passes through all three points. In this problem, we show that these three points are collinear if and only if $(s - t)/(u - t)$ is real.

Let O be the origin. Consider the translation that maps T to the origin. Let S' and U' be the images of S and U, respectively, under this translation, and let the complex numbers s' and u' correspond to S' and U', respectively.

(a) Explain why S, T, and U are collinear if and only if S', O, and U' are. What must be true of s' and u' if and only if S', O, and U' are collinear?

(b) Show that S, T, and U are collinear if and only if $(s - t)/(u - t)$ is real.

Problem 8.11: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b - a)/(d - c)$ is real.

Problem 8.12: Use complex numbers to show that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Problem 8.13: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. Find a condition similar to that in Problem 8.11 that is necessary and sufficient to show that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Problem 8.14: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments.

(a) Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(a-b)(\overline{c}-\overline{d}) - (\overline{a}-\overline{b})(c-d) = 0$.

(b) Show that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $(a-b)(\overline{c}-\overline{d}) + (\overline{a}-\overline{b})(c-d) = 0$.

Problem 8.15: Suppose $VWXYZ$ is a pentagon such that the altitudes from V, W, X, and Y to the respective opposite sides of the pentagon (V to $\overline{XY}$, W to $\overline{YZ}$, etc.) meet at a common point P. In this problem, we show that the altitude from Z to $\overline{WX}$ passes through P as well.

(a) Write four equations corresponding to the four given altitudes. (Tip: Choose your origin wisely!)

(b) Write an equation that must be true if the altitude from Z to $\overline{WX}$ passes through P.

(c) Combine your equations in part (a) to produce the equation in part (b).

Problem 8.9: Let W and Z be points in the complex plane corresponding to nonzero complex numbers w and z, respectively, and let O be the origin. Show that $\overleftrightarrow{WZ}$ passes through O if and only if z/w is a real number.

Solution for Problem 8.9: We must show both of the following:

(a) If $\overleftrightarrow{WZ}$ passes through O, then z/w is real.

(b) If z/w is real, then $\overleftrightarrow{WZ}$ passes through O.

Important: Suppose $\mathcal{P}$ and $\mathcal{Q}$ are mathematical statements. The statement "$\mathcal{P}$ if and only if $\mathcal{Q}$" means both of the following:

- If $\mathcal{P}$ is true, then $\mathcal{Q}$ is also true.
- If $\mathcal{Q}$ is true, then $\mathcal{P}$ is also true.

Both of these statements must be proved in order to show that "$\mathcal{P}$ if and only if $\mathcal{Q}$."

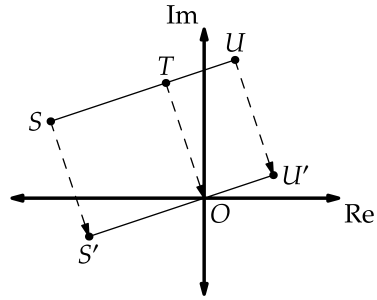
We'll prove parts (a) and (b) separately. We'll start with part (a). Since W and Z are on the same line through the origin, we know that we can transform W to Z by performing a dilation about the origin (possibly with a negative scale factor). Therefore, there is some real constant c such that $z = cw$, from which we have $\frac{z}{w} = c$.

Next we must take care of part (b). If $\frac{z}{w} = c$, where c is a real number, then we have $z = cw$. Since c is real, this means that Z is the image of W under some dilation about the origin, which means that Z is on $\overleftrightarrow{OW}$, as desired.

We also could have tackled part (b) by noting that if $w = re^{i\theta}$, then $z = cre^{i\theta}$. If $c > 0$, then W and Z have the same argument. If $c < 0$, then their arguments differ by π. In both cases, W and Z lie on the same line through the origin. □

We say that three points are **collinear** if they lie on the same line. Now that we know how to tell if the line through two given points in the complex plane passes through the origin, let's see if we can develop a similar test to determine if any given three points are collinear.

Problem 8.10: Suppose S, T, and U are distinct points in the complex plane corresponding to complex numbers s, t, and u, respectively. Show that S, T, and U are collinear if and only if $(s-t)/(u-t)$ is real.



Solution for Problem 8.10: Just as we did with the general rotation problem in Problem 8.7, we tackle the general problem of collinearity by using translations to turn it into a problem we already know how to solve. We know how to tell if the line through two points passes through the origin, so we translate all three points such that the image of one of these points is the origin. The images of S, T, and U are collinear if and only if S, T, and U are. (We can see this by noting that applying a translation to $\angle STU$ will not change the measure of the angle.)

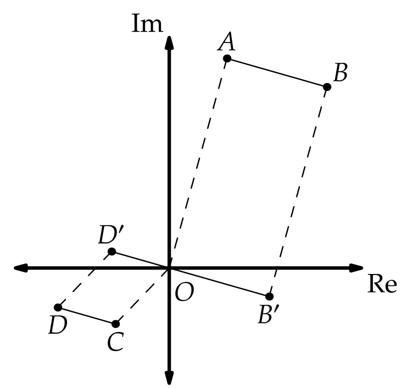
The expressions $s - t$ and $u - t$ in $(s - t)/(u - t)$ suggest translating T to the origin, since such a translation maps S and U to points S' and U' such that $s' = s - t$ and $u' = u - t$. Letting O be the origin, we can now apply our result from Problem 8.9. There, we showed that $\overleftrightarrow{S'U'}$ passes through the origin if and only if s'/u' is real. Substituting our expressions for s' and u', we see that $\overleftrightarrow{S'U'}$ passes through the origin if and only if $(s - t)/(u - t)$ is a real number. Since S', O, and U' are collinear if and only if $(s - t)/(u - t)$ is real, and S, T, and U are collinear if and only if S', O, and U' are collinear, we conclude that S, T, and U are collinear if and only if $(s - t)/(u - t)$ is real. □

Important: The diagram at right above is not the only possible configuration; it is possible that T is on $\overleftrightarrow{SU}$, but not between S and U. However, the fact that $\overleftrightarrow{S'U'}$ passes through the origin if and only if $(s - t)/(u - t)$ is real *does not depend upon the possible orderings of S', U', and O.* So, we don't have to worry about different cases.

Important: Suppose S, T, and U are distinct points in the complex plane corresponding to complex numbers s, t, and u, respectively. S, T, and U are collinear if and only if $(s - t)/(u - t)$ is real.

We'll now turn to parallel lines. In addition to the usual definition that two lines are parallel if they do not intersect, we will also say that a line is parallel to itself. We can think of this as "all lines with the same direction are parallel." We will make the concept of "direction" of a line more rigorous in Section 9.3.

Problem 8.11: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b - a)/(d - c)$ is a real number.



Solution for Problem 8.11: There's no need to reinvent the wheel here; we have a pretty clear guide how to tackle this problem in our solution to Problem 8.10. We start by translating $\overline{AB}$ such that the image of A is the origin, O. Let the image of B under this translation be B', so that $b' = b - a$. Similarly, we translate $\overline{CD}$ such that the image of C is the origin and the image of D is D', which gives us $d' = d - c$. Since $\overleftrightarrow{OB'} \parallel \overleftrightarrow{AB}$ and $\overleftrightarrow{OD'} \parallel \overleftrightarrow{CD}$, we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $\overleftrightarrow{OB'}$ and $\overleftrightarrow{OD'}$ are the same line, because there is only one line through O parallel to $\overleftrightarrow{AB}$. In Problem 8.9, we saw that $\overleftrightarrow{OB'}$ and $\overleftrightarrow{OD'}$ are the same line if and only if b'/d' is real. Substituting our expressions above for b' and d', this tells us that $\overleftrightarrow{OB'}$ and $\overleftrightarrow{OD'}$ are the same line if and only if $(b - a)/(d - c)$ is real. Every step in our reasoning is an "if and only if" step, so we have proved that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b - a)/(d - c)$ is a real number. □

Important: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. Then lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel if and only if $(b - a)/(d - c)$ is a real number.

Problem 8.12: Use complex numbers to show that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Solution for Problem 8.12: Let complex numbers a, b, c, and d correspond to the vertices of quadrilateral $ABCD$ in the complex plane. Then, the midpoint of diagonal $\overline{AC}$ is $\frac{a+c}{2}$ and the midpoint of diagonal $\overline{BD}$ is $\frac{b+d}{2}$.

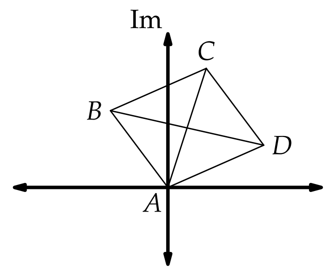
First, we show that if the diagonals of $ABCD$ bisect each other, then $ABCD$ is a parallelogram. If the diagonals of $ABCD$ bisect each other, then the midpoints of $\overline{AC}$ and $\overline{BD}$ are the same. That is, we have

$$\frac{a+c}{2} = \frac{b+d}{2},$$

so $a + c = b + d$. We'd like to show that $\overline{AB} \parallel \overline{DC}$, which we can prove by showing that $(b - a)/(c - d)$ is real. So, we rearrange $a + c = b + d$ as $c - d = b - a$. Dividing both sides by $c - d$ gives $\frac{b-a}{c-d} = 1$. Since this is real, we have $\overline{AB} \parallel \overline{DC}$. We could go through essentially the same steps to show that $\overline{AD} \parallel \overline{BC}$, or we could note that since $c - d = b - a$, we have $|c - d| = |b - a|$, which means that $DC = AB$. Combining $AB = DC$ with $\overline{AB} \parallel \overline{DC}$ tells us that $ABCD$ is a parallelogram.

Important: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. We have $AB = CD$ if and only if $|b - a| = |d - c|$.

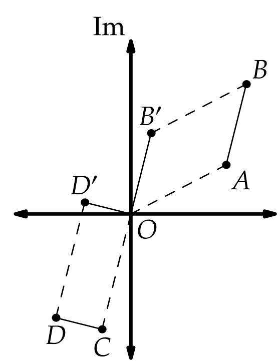
Next, we show that if $ABCD$ is a parallelogram, then the diagonals bisect each other. We can take a bit of a shortcut here by letting A be the origin, which means the midpoint of $\overline{AC}$ is simply $\frac{c}{2}$. Then, because $\overline{AB} \parallel \overline{DC}$ and $AB = DC$, the translation that maps B to A also maps C to D. This translation maps B to the origin, so it maps any complex number z to $z - b$. Because this translation also maps C to D, we have $d = c - b$. Therefore, we have $d + b = c$, so the midpoint of $\overline{BD}$ is $\frac{b+d}{2} = \frac{c}{2}$. This tells us that diagonals $\overline{AC}$ and $\overline{BD}$ have the same midpoint, which means they bisect each other. □



Having tackled parallel lines, let's turn to perpendicular lines.

Problem 8.13: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. Find a condition similar to that in Problem 8.11 that is necessary and sufficient to show that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

Solution for Problem 8.13: Solution 1: Use the method of previous problems. As before, we start with translations to the origin. We translate $\overline{AB}$ such that the image of A is the origin. This translation maps B to a point B' that corresponds to $b' = b - a$. Similarly, we translate $\overline{CD}$ such that the image of C is the origin, which means the image of D is the point D' that corresponds to $d' = d - c$.



Translating a segment does not change its orientation, so $\overleftrightarrow{AB} \parallel \overleftrightarrow{OB'}$. Therefore, the angle $\overleftrightarrow{OD'}$ makes with $\overleftrightarrow{OB'}$ is the same as the angle $\overleftrightarrow{OD'}$ makes with $\overleftrightarrow{AB}$. Similarly, $\overleftrightarrow{OB'}$ makes the same angle with $\overleftrightarrow{CD}$ as it makes with $\overleftrightarrow{OD'}$. Combining these observations, we see that the angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ is the same as that between $\overline{OB'}$ and $\overline{OD'}$. This means that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $\overline{OB'} \perp \overline{OD'}$. Therefore, our problem now is to prove that $\overline{OB'} \perp \overline{OD'}$ if and only if $\frac{b-a}{d-c}$ is imaginary. Since $b' = b - a$ and $d' = d - c$, we must show that $\overline{OB'} \perp \overline{OD'}$ if and only if b'/d' is imaginary.

If $\overline{OB'} \perp \overline{OD'}$, then the arguments of b' and d' differ by $\frac{\pi}{2}$. Therefore, if d' in exponential form is $re^{i\theta}$, then b' is $se^{i(\theta+(\pi/2))}$ or $se^{i(\theta-(\pi/2))}$, where r and s are real numbers. If $b' = se^{i(\theta+(\pi/2))}$, then we have

$$\frac{b'}{d'} = \frac{se^{i(\theta+(\pi/2))}}{re^{i\theta}} = \frac{s}{r}e^{\pi i/2} = \frac{s}{r}i.$$

Similarly, if $b' = se^{i(\theta-(\pi/2))}$, then we find $\frac{b'}{d'} = -\frac{s}{r}i$. In either case, $\frac{s}{r}$ is real, so $\frac{b'}{d'}$ is an imaginary number.

We must also show that if $\frac{b'}{d'} = ki$ for some real constant k, then $\overline{OB'} \perp \overline{OD'}$. Multiplying both sides by d' gives $b' = (kd')i$. Therefore, B' is a 90° counterclockwise rotation about the origin of the point corresponding to kd'. The point corresponding to kd' is a dilation of D' about the origin, so it is on $\overleftrightarrow{OD'}$. Since B' is a 90° rotation about the origin of a point on $\overleftrightarrow{OD'}$, we have $\overline{OB'} \perp \overline{OD'}$, as desired.

Solution 2: Use the result of Problem 8.11. We turn our problem about perpendicular lines into one about parallel lines by rotating one of the lines 90° about the origin. Let $\overleftrightarrow{C'D'}$ be the image of $\overleftrightarrow{CD}$ upon a 90° rotation counterclockwise about the origin. Therefore, we have $c' = ci$ and $d' = di$. We have $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $\overleftrightarrow{AB} \parallel \overleftrightarrow{C'D'}$. From Problem 8.11, we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{C'D'}$ if and only if $\frac{b-a}{d'-c'}$ is a real number. That is, $\overleftrightarrow{AB} \parallel \overleftrightarrow{C'D'}$ if and only if $\frac{b-a}{d'-c'} = t$ for some real constant t. Substituting our expressions for c' and d', this equation becomes

$$\frac{b-a}{di-ci} = t.$$

Multiplying both sides by i gives $\frac{b-a}{d-c} = ti$, which is imaginary. Therefore, we have $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $\frac{b-a}{d-c}$ is an imaginary number, as before. □

Important: Let $\overline{AB}$ and $\overline{CD}$ be segments in the complex plane, and let a, b, c, and d be complex numbers corresponding to the endpoints of these segments. Then $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $(b-a)/(d-c)$ is an imaginary number.

Problem 8.14: Let A, B, C, and D be four different points in the complex plane, and let a, b, c, and d be complex numbers corresponding to these points.

(a) Show that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b-a)(\overline{d}-\overline{c}) - (\overline{b}-\overline{a})(d-c) = 0$.

(b) Show that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $(b-a)(\overline{d}-\overline{c}) + (\overline{b}-\overline{a})(d-c) = 0$.

Solution for Problem 8.14:

(a) In Problem 8.11, we showed that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b-a)/(d-c)$ is real. Back on page 204, we showed that a complex number z is real if and only if $\overline{z} = z$, which we can write as $z - \overline{z} = 0$. So, because $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b-a)/(d-c)$ is real, we must have $z - \overline{z} = 0$ for $z = (b-a)/(d-c)$:

$$\frac{b-a}{d-c} - \overline{\left(\frac{b-a}{d-c}\right)} = 0. \tag{8.1}$$

Since we have

$$\overline{\left(\frac{b-a}{d-c}\right)} = \frac{\overline{b-a}}{\overline{d-c}} = \frac{\overline{b}-\overline{a}}{\overline{d}-\overline{c}},$$

we can write Equation (8.1) as

$$\frac{b-a}{d-c} - \frac{\overline{b}-\overline{a}}{\overline{d}-\overline{c}} = 0.$$

Multiplying both sides by $d-c$ and by $\overline{d}-\overline{c}$, we have the desired

$$(b-a)(\overline{d}-\overline{c}) - (\overline{b}-\overline{a})(d-c) = 0.$$

(b) We can follow essentially the same steps as in the previous part. In Problem 8.13 we showed that $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $(b-a)/(d-c)$ is imaginary. Back on page 204, we learned that a complex number z is imaginary if and only if $\overline{z} = -z$, which we can rearrange as $z + \overline{z} = 0$. So, letting $z = (b-a)/(d-c)$, we have

$$\frac{b-a}{d-c} + \overline{\left(\frac{b-a}{d-c}\right)} = 0,$$

which we can write as

$$\frac{b-a}{d-c} + \frac{\overline{b}-\overline{a}}{\overline{d}-\overline{c}} = 0.$$

Multiplying both sides by $(d-c)(\overline{d}-\overline{c})$ gives

$$(b-a)(\overline{d}-\overline{c}) + (\overline{b}-\overline{a})(d-c) = 0.$$

□

Our work in Problem 8.14 allows us to write the conditions "$(b-a)/(d-c)$ is a real number" and "$(b-a)/(d-c)$ is an imaginary number" as equations involving only a, b, c, and d. These equations are often much more convenient to work with than the qualitative description of $(b-a)/(d-c)$ being real or imaginary.

Important: Let A, B, C, and D be four different points in the complex plane, and let a, b, c, and d be complex numbers corresponding to these points.

- We have $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ if and only if $(b-a)(\overline{d}-\overline{c}) - (\overline{b}-\overline{a})(d-c) = 0$.
- We have $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if $(b-a)(\overline{d}-\overline{c}) + (\overline{b}-\overline{a})(d-c) = 0$.

These aren't really new observations; they are equations that follow from the conditions we found earlier under which $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are either parallel or perpendicular.

Looking back at our solution to Problem 8.14, we actually proved something a little bit *stronger* than we were required to. We needed A and B to be different to define line $\overleftrightarrow{AB}$, and needed C and D be different to define $\overleftrightarrow{CD}$. But nowhere in our proof do we use the fact that A and B differ from C and D! We could have had B and D be the same point, and every step in the proof holds. Therefore, by letting B and D be the same point, we have a condition for A, B, and C to be collinear:

Important: Let A, B, and C be three different points in the complex plane, and let a, b, and c be complex numbers corresponding to these points. A, B, and C are collinear if and only if

$$(b-a)(\overline{b}-\overline{c}) - (\overline{b}-\overline{a})(b-c) = 0.$$

The condition for perpendicularity given above also holds if we let B and D be the same point.

Concept: Sometimes we can prove a statement by proving a more general statement.

Problem 8.15: Suppose $VWXYZ$ is a pentagon such that the altitudes from V, W, X, and Y to the respective opposite sides of the pentagon (V to $\overline{XY}$, W to $\overline{YZ}$, etc.) meet at a common point P. Show that the altitude from Z to $\overline{WX}$ passes through P as well.

Solution for Problem 8.15: We are given four pairs of perpendicular lines and must use them to prove a fifth pair of lines are perpendicular. One point is common to all the pairs of lines: P. So, we'll let P be the origin, expecting that to make our algebra simpler. We let v, w, x, y, z be the complex numbers corresponding to the respective vertices of the pentagon. Since $\overleftrightarrow{VP} \perp \overleftrightarrow{XY}$, we have

$$(v-0)(\overline{x}-\overline{y}) + (\overline{v}-0)(x-y) = 0.$$

We simplify $v-0$ and $\overline{v}-0$, and then write similar equations for each of the other three altitudes, and we have

$$\begin{aligned} v(\overline{x}-\overline{y}) + \overline{v}(x-y) &= 0,\\ w(\overline{y}-\overline{z}) + \overline{w}(y-z) &= 0,\\ x(\overline{z}-\overline{v}) + \overline{x}(z-v) &= 0,\\ y(\overline{v}-\overline{w}) + \overline{y}(v-w) &= 0. \end{aligned}$$

We'd like to prove that $\overleftrightarrow{ZP} \perp \overleftrightarrow{WX}$, which is true if and only if

$$z(\overline{w}-\overline{x}) + \overline{z}(w-x) = 0. \tag{8.2}$$

Looking at our initial four equations above, we see that all the terms in the expansion of Equation (8.2) appear in the expansions of the first four equations. Unfortunately, there are a bunch more terms in those four equations. Let's go ahead and expand them and get a closer look at those extra terms:

$$\begin{aligned} v\overline{x} - v\overline{y} + \overline{v}x - \overline{v}y &= 0,\\ w\overline{y} - w\overline{z} + \overline{w}y - \overline{w}z &= 0,\\ x\overline{z} - x\overline{v} + \overline{x}z - \overline{x}v &= 0,\\ y\overline{v} - y\overline{w} + \overline{y}v - \overline{y}w &= 0. \end{aligned}$$

If we add all of these equations, the extra terms will all cancel! For example, the $v\overline{x}$ in the first equation cancels with the $-\overline{x}v$ in the third equation, and the $-v\overline{y}$ in the first equation cancels with the $\overline{y}v$ in the fourth equation. The terms that don't cancel when we add all four equations leave us with

$$x\overline{z} - w\overline{z} + \overline{x}z - \overline{w}z = 0.$$

Factoring, multiplying by -1, and doing a little rearranging, gives us

$$z(\overline{w}-\overline{x}) + \overline{z}(w-x) = 0,$$

so $\overleftrightarrow{ZP} \perp \overleftrightarrow{WX}$, as desired. □

We glossed over an important strategy in setting up our solution to Problem 8.15. We wrote the four equations corresponding to the four given altitudes in a way that made it clear that we would have a lot of cancellation when we added the four equations. For example, if instead of writing the four equations

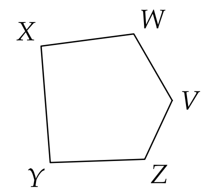
$$\begin{aligned} v(\overline{x}-\overline{y}) + \overline{v}(x-y) &= 0,\\ w(\overline{y}-\overline{z}) + \overline{w}(y-z) &= 0,\\ x(\overline{z}-\overline{v}) + \overline{x}(z-v) &= 0,\\ y(\overline{v}-\overline{w}) + \overline{y}(v-w) &= 0, \end{aligned}$$

we had written the four equations

$$\begin{aligned} v(\overline{x}-\overline{y}) + \overline{v}(x-y) &= 0,\\ w(\overline{y}-\overline{z}) + \overline{w}(y-z) &= 0,\\ x(\overline{z}-\overline{v}) + \overline{x}(z-v) &= 0,\\ y(\overline{w}-\overline{v}) + \overline{y}(w-v) &= 0, \end{aligned}$$

then adding the four equations won't cancel the $v\overline{y}$ and $y\overline{v}$ terms. We'd have to subtract the last equation instead, and our key step would be "add the first three equations and subtract the last," which is a much harder key step to see. (Look at the last equation in each set to see the difference between the two systems.)

We make sure our equations are set up in a convenient way by building them in a consistent manner. Here, we can set our equations up with a nice symmetry by visualizing the pentagon, and making sure the three points in each equation appear in the same order that they appear when we travel counterclockwise around the pentagon. For example, we go counterclockwise when we go from V to X to Y around the pentagon, and v, x, and y appear in that order in the first equation. Similarly, we go counterclockwise to go from W to Y to Z, so w, y, z appear in that order in the second equation, and so on.



Concept: Maintaining symmetry in systems of equations helps us find effective ways to work with the equations.

Exercises

8.2.1 Which of the following complex numbers lies on the line through $1 + 3i$ and $5 - 2i$ in the complex plane: $3 - 2i$, $9 - 7i$, $-2 + 5i$.

8.2.2 Show that the equation whose graph consists of all complex numbers z on the line through a and b in the complex plane is

$$\frac{z-a}{b-a} = \frac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}.$$

8.2.3 Prove that the quadrilateral formed by connecting the midpoints of any quadrilateral is a parallelogram.

8.2.4 In Problem 8.13, we used an algebraic approach to show that if O is the origin and points B' and D' in the complex plane satisfy $\overline{OB'} \perp \overline{OD'}$, then $\frac{b'}{d'}$ is an imaginary number, where b' and d' are the complex numbers corresponding to B' and D', respectively. Prove this fact using geometric transformations. That is, find a sequence of transformations that maps B' to D', and use this sequence to explain why $\frac{b'}{d'}$ must be imaginary.

8.2.5 A **median** of a triangle is the segment from a vertex of the triangle to the midpoint of the opposite side. In this problem, we show that the medians of a triangle are concurrent at a point called the **centroid**, and show that the distance from a vertex of the triangle to the triangle's centroid is 2/3 the length of the median from that vertex.

(a) Let a, b, and c be complex numbers corresponding to the respective vertices of $\triangle ABC$. Show that if $g = (a + b + c)/3$ corresponds to point G, then point G is on all three medians.

(b) Let M be the midpoint of $\overline{BC}$. Show that $AG/GM = 2$.

8.2.6 Show that the complex numbers a, b, and c are collinear if and only if

$$a\overline{b} + b\overline{c} + c\overline{a} = \overline{a}b + \overline{b}c + \overline{c}a.$$

8.2.7 In convex quadrilateral $WXYZ$, the midpoints of $\overline{WX}$, $\overline{WY}$, and $\overline{XZ}$ are collinear. Show that the line through these points also passes through the midpoint of $\overline{YZ}$.

8.2.8★ Let A correspond to $1 - 5i$, B correspond to $3 + 2i$, C correspond to $-7\sqrt{3} + 7i$, and D correspond to $-2 - 2i\sqrt{3}$. Find the acute angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

8.3 Distance

We introduced the magnitude of a complex number as the distance between that complex number and the origin in the complex plane. From this definition, we also saw that if w and z are complex numbers, then $|w - z|$ is the