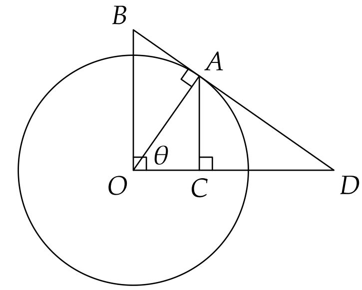
4.1.4 Find the area of $\triangle ABC$ if $AB = 4BC = 12$ and $\angle ABC = 30°$.

4.1.5 I'm standing at the peak of a mountain that is 14,000 feet above sea level. The angle of depression from this peak to a nearby smaller peak is 4°. On my map, these two peaks are represented by points that are 1 inch apart. If each inch on my map represents 1.2 miles, and there are 5280 feet in a mile, then how many feet above sea level is the second peak?

4.1.6 The radius of the circle at right is 1. For each of the following trigonometric expressions, find a segment in the diagram that has length equal to the trigonometric expression: $\sin\theta$, $\cos\theta$, $\sec\theta$, $\csc\theta$, $\tan\theta$, $\cot\theta$. (Note: You are *not* asked to express each trig function in terms of multiple segments in the diagram; you must find a segment whose length *equals* the corresponding trig function.)



4.1.7 A, B, and C are vertices of a cube such that $\overline{AB}$ is an interior diagonal of the cube and $\overline{AC}$ is a diagonal of a face of the cube. Find $\cos\angle ABC$.

4.1.8★ Square $ABCD$ has center O and $AB = 900$. Points E and F are on $\overline{AB}$ with $AE < BF$ and E between A and F such that $\angle EOF = 45°$ and $EF = 400$. Find BF. *(Source: AIME)*

4.2 Law of Cosines

Problems

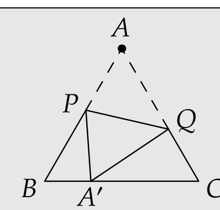
Problem 4.8: In $\triangle ABC$, let $AC = 14$, $BC = 12$, and $\angle C = 34°$. In this problem, we find AB to the nearest hundredth.
(a) Draw altitude $\overline{BX}$ from B to $\overline{AC}$. Find CX and BX to the nearest hundredth.
(b) Find XA to the nearest hundredth.
(c) Use parts (a) and (b) to find AB to the nearest hundredth.

Problem 4.9: Let $\triangle ABC$ be an acute triangle with $a = BC$, $b = AC$, and $c = AB$. Use Problem 4.8 as a guide to prove that
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Problem 4.10: An airplane leaves an aircraft carrier and flies due south at 400 km/h. The carrier proceeds 60° east of north at 32 km/h. If the plane has enough fuel for 5 hours of flying, what is the maximum distance south the pilot can travel, so that the fuel remaining will allow a safe return to the carrier? (You may assume Earth is flat in this problem.) *(Source: CEMC)*

Problem 4.11: In $\triangle ABC$, we have $AB = 5$, $BC = 7$, and $AC = 8$. Find $\angle BAC$.

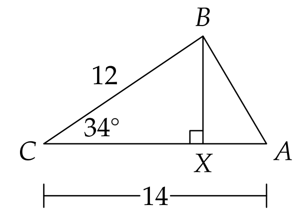
Problem 4.12: Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$, then what is the length of crease $\overline{PQ}$? *(Source: AMC 12)*
(a) Why do 60° and 120° angles suggest trying the Law of Cosines?
(b) Find PQ. **Hints:** 228



Problem 4.13: In quadrilateral $ABCD$, we have $\angle A = \angle C$, $AB = CD = 180$, and $AD \neq BC$. The perimeter of $ABCD$ is 640. Find $\cos A$. *(Source: AIME)* **Hints:** 175

Problem 4.8: In $\triangle ABC$, let $AC = 14$, $BC = 12$, and $\angle C = 34^\circ$. Find AB to the nearest hundredth.

Solution for Problem 4.8: We know how to use trigonometry to find side lengths in right triangles, so we start by drawing an altitude from B to $\overline{AC}$ as shown. This creates right triangle $\triangle BCX$ with the 34° angle as one of its acute angles. From right triangle $\triangle CBX$, we have



$$\frac{BX}{BC} = \sin C \approx 0.559,$$

so $BX \approx 0.559(BC) \approx 6.71$. Similarly, we have $CX/BC = \cos C \approx 0.829$, so $CX \approx 9.95$. This doesn't tell us AB yet, but we now have the length of one leg of $\triangle BXA$. If we can find the other, we can use the Pythagorean Theorem to find AB. Fortunately, XA is easy to find: $XA = AC - CX \approx 4.05$. Now, we can use the Pythagorean Theorem to find

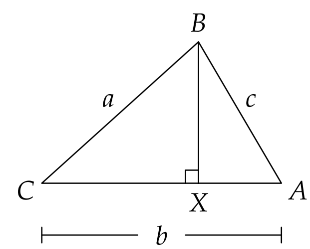
$$AB = \sqrt{BX^2 + XA^2} \approx 7.84.$$

□

In Problem 4.8, we were given two side lengths of a triangle and the measure of the angle between these two sides. We then found the third side. There was nothing particularly special about the side lengths or the angle. We might be able to follow essentially the same process for any triangle. Let's give it a try.

Problem 4.9: Let $\triangle ABC$ be an acute triangle with $a = BC$, $b = AC$, and $c = AB$. Find a formula for c in terms of a, b, and $\angle C$.

Solution for Problem 4.9: We use Problem 4.8 as a guide. In fact, this problem is essentially the same as Problem 4.8, but with variables a, b, and $\angle C$ in place of the numbers that were given in the earlier problem. We can use the same steps. We draw altitude $\overline{BX}$ from B to $\overline{AC}$. Then, we have $\sin C = BX/BC$, so $BX = BC \sin C = a \sin C$. We also have $\cos C = CX/BC$, so $CX = BC \cos C = a \cos C$. Therefore, we have $AX = AC - CX = b - a\cos C$. Next we apply the Pythagorean Theorem to $\triangle ABX$ to find $AB^2 = BX^2 + AX^2$. Substituting our expressions for these three sides gives us



$$\begin{aligned} c^2 &= a^2 \sin^2 C + (b - a\cos C)^2 \\ &= a^2 \sin^2 C + b^2 - 2ab\cos C + a^2\cos^2 C \\ &= a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab\cos C. \end{aligned}$$

Since $\sin^2 C + \cos^2 C = 1$, we have

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

□

We can test our new formula with the data from Problem 4.8.

Concept: Once you think you've found a new formula, check your work by trying the formula on specific examples you have solved without the formula.

In Problem 4.8, we have $a = 12$, $b = 14$, and $\angle C = 34^\circ$, so we have

$$c^2 = a^2 + b^2 - 2ab\cos C \approx 61.44.$$

Taking the square root of both sides gives $c \approx 7.84$, which agrees with our answer from Problem 4.8.

With a little more casework (which you'll supply as an Exercise), we can show that this equation holds for any triangle ABC.

Important: Let $a = BC$, $b = AC$, and $c = AB$ in $\triangle ABC$. The **Law of Cosines** states that

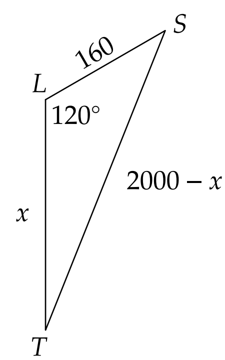
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Notice that when $\angle C = 90^\circ$, we have $\cos C = 0$, so the Law of Cosines becomes $c^2 = a^2 + b^2$, which is just the Pythagorean Theorem.

Problem 4.8 is just a specific example of Problem 4.9, so we call Problem 4.9 a **generalization** of Problem 4.8.

Concept: Solving a specific example of a general problem can often provide a guide for solving the general problem.

Problem 4.10: An airplane leaves an aircraft carrier and flies due south at 400 km/h. The carrier proceeds 60° east of north at 32 km/h. If the plane has enough fuel for 5 hours of flying, what is the maximum distance south the pilot can travel, so that the fuel remaining will allow a safe return to the carrier? (You may assume Earth is flat in this problem.) *(Source: CEMC)*



Solution for Problem 4.10: We start with a diagram, including the path of the ship and the path of the plane. The plane leaves the ship at L, flies south to T before turning, and then flies to S, where it lands on the ship. Because the plane flies south and the ship goes 60° east of north, we have $\angle TLS = 180^\circ - 60^\circ = 120^\circ$. If the plane flies for 5 hours, then the ship moves $32 \cdot 5 = 160$ km and the plane flies $400 \cdot 5 = 2000$ km. Therefore, $LS = 160$ and $LT + TS = 2000$. We let the desired distance, LT, be x, so $TS = 2000 - x$, as shown. We have expressions for all three sides of $\triangle LST$, and we know one angle, so we apply the Law of Cosines:

$$TS^2 = LT^2 + LS^2 - 2(LT)(LS)\cos\angle TLS.$$

Inserting our expressions and values for lengths and the angle, we have

$$(2000 - x)^2 = x^2 + 160^2 - 2(x)(160)\cos 120^\circ.$$

We have $\cos 120^\circ = -\frac{1}{2}$ (which is one reason the Law of Cosines is often helpful in problems involving 120° angles), and expanding the left side of the equation gives

$$2000^2 - 4000x + x^2 = x^2 + 160^2 + 160x.$$

Solving this equation gives $x \approx 955$, so the plane can fly approximately 955 km south. □

Problem 4.11: In $\triangle ABC$, we have $AB = 5$, $BC = 7$, and $AC = 8$. Find $\angle BAC$.

Solution for Problem 4.11: We are given all three side lengths of the triangle, so we have all the information in the Law of Cosines except the angle measure. Therefore, we can solve for $\cos\angle BAC$ with the Law of Cosines. We have

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos\angle BAC,$$

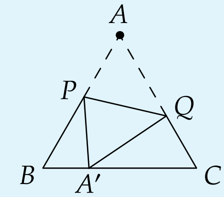
so

$$\cos\angle BAC = \frac{AB^2 + AC^2 - BC^2}{2(AB)(AC)} = \frac{25 + 64 - 49}{2(5)(8)} = \frac{1}{2}.$$

Since $0^\circ < \angle BAC < 180^\circ$, $\cos\angle BAC = \frac{1}{2}$ gives us $\angle BAC = 60^\circ$. □

Concept: The Law of Cosines can be used to find angle measures as well as side lengths.

Problem 4.12: Equilateral triangle ABC has been creased and folded so that vertex A now rests at A' on $\overline{BC}$ as shown. If $BA' = 1$ and $A'C = 2$, then what is the length of crease $\overline{PQ}$? *(Source: AMC 12)*



Solution for Problem 4.12: We know that the side length of the original equilateral triangle is $BA' + A'C = 3$, but we can't immediately find any other lengths. So, we assign a variable and hope to express other side lengths in terms of that variable. We might start by letting $z = PQ$, but we don't see any way to express another side length in terms of z. Therefore, we start by letting $x = BP$, which gives us $AP = AB - BP = 3 - x$. But now we seem stuck.

Concept: When stuck on a problem, focus on information you haven't used yet.

We haven't used the fact that $\triangle PA'Q$ is the reflection of $\triangle PAQ$ over $\overline{PQ}$. This tells us that $PA' = PA$, so $PA' = 3 - x$. Now, we have expressions for the lengths of all three sides of $\triangle PA'B$. We also know that $\angle PBA' = 60^\circ$, so we apply the Law of Cosines:

$$(PA')^2 = BP^2 + (BA')^2 - 2(BP)(BA')\cos\angle PBA',$$

so

$$(3 - x)^2 = x^2 + 1 - 2x\cos 60^\circ.$$

Since $\cos 60^\circ = \frac{1}{2}$, we have $(3 - x)^2 = x^2 - x + 1$. Expanding the left side gives $9 - 6x + x^2 = x^2 - x + 1$, from which we find $x = \frac{8}{5}$. Therefore, we have $BP = \frac{8}{5}$ and $PA' = PA = 3 - BP = \frac{7}{5}$.

We can do the same thing with $\overline{AC}$!

Concept: If a tactic gives some information in a problem, but doesn't completely solve it, then try applying that tactic to the problem again in another way.

We let $QC = y$, so $A'Q = AQ = 3 - y$. Applying the Law of Cosines to $\triangle A'QC$ gives

$$(A'Q)^2 = (A'C)^2 + QC^2 - 2(A'C)(QC)\cos\angle A'CQ,$$

so

$$(3 - y)^2 = 4 + y^2 - 2(2)(y)\cos 60^\circ.$$

Solving this equation for y gives us $y = \frac{5}{4}$, so $QC = \frac{5}{4}$ and $A'Q = AQ = 3 - QC = \frac{7}{4}$.

One more time! We have $AP = \frac{7}{5}$, $AQ = \frac{7}{4}$, and $\angle PAQ = 60^\circ$, so we apply the Law of Cosines to find

$$\begin{aligned} PQ^2 = AP^2 + AQ^2 - 2(AP)(AQ)\cos\angle PAQ &= \left(\frac{7}{5}\right)^2 + \left(\frac{7}{4}\right)^2 - 2\left(\frac{7}{5}\right)\left(\frac{7}{4}\right)\left(\frac{1}{2}\right) \\ &= \frac{49}{25} + \frac{49}{16} - \frac{49}{20} = \frac{49\cdot 16 + 49\cdot 25 - 49\cdot 20}{400} = \frac{49\cdot 21}{400}. \end{aligned}$$

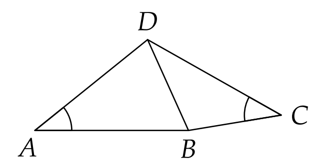
Taking the square root gives $PQ = 7\sqrt{21}/20$. □

Each of the previous three problems featured a 60° or 120° angle. Usually, when we see 60° or 120°, our first strategy is to look for equilateral triangles or 30-60-90 triangles. But since $\cos 60^\circ = \frac{1}{2}$ and $\cos 120^\circ = -\frac{1}{2}$, the Law of Cosines has a particularly simple form when the angle is 60° or 120°.

Concept: If a geometry problem contains a 60° or 120° angle, you may be able to tackle it with the Law of Cosines.

WARNING!! The Law of Cosines can lead to some pretty heavy algebra, so usually we look for equilateral triangles or 30-60-90 triangles before pulling out the Law of Cosines on a problem that involves 60° or 120° angles.

Problem 4.13: In quadrilateral $ABCD$, we have $\angle A = \angle C$, $AB = CD = 180$, and $AD \neq BC$. The perimeter of $ABCD$ is 640. Find $\cos A$. *(Source: AIME)*



Solution for Problem 4.13: That the question asks for the cosine of an angle suggests that we either build right triangles or we apply the Law of Cosines. But we don't have any triangles. So, we make some. The obvious candidate for building triangles is to draw diagonals. We don't want to draw diagonal $\overline{AC}$, since that will break up two angles that we know are equal, and we'll probably have to use that angle equality. So, we try drawing $\overline{BD}$ first, thereby creating two triangles that share a side and have equal angles. We apply the Law of Cosines to each, using $\angle A$ in $\triangle ABD$ and $\angle C$ in $\triangle CBD$:

$$BD^2 = AB^2 + AD^2 - 2(AB)(AD)\cos A,$$
$$BD^2 = CB^2 + CD^2 - 2(CB)(CD)\cos C.$$

Setting our two expressions for BD^2 equal, and noting that $\cos C = \cos A$ because $\angle A = \angle C$, gives us

$$AB^2 + AD^2 - 2(AB)(AD)\cos A = CB^2 + CD^2 - 2(CB)(CD)\cos A.$$

We also have $AB = CD = 180$, so $AB^2 = CD^2$, and the equation is now

$$AD^2 - 2(180)(AD)\cos A = CB^2 - 2(180)(CB)\cos A.$$

To solve for $\cos A$, we group the $\cos A$ terms on the left, and we have

$$2(180)(CB)\cos A - 2(180)(AD)\cos A = CB^2 - AD^2,$$

so

$$\cos A = \frac{CB^2 - AD^2}{2(180)(CB - AD)}.$$

(This step only avoids dividing by 0 because we are told that $AD \neq BC$. This gives us some confidence that we're on the right track.) Factoring the numerator of the right side as a difference of squares, we have

$$\cos A = \frac{(CB - AD)(CB + AD)}{2(180)(CB - AD)} = \frac{CB + AD}{2(180)}.$$

Now, we just have to find $CB + AD$. We go back to the problem, and look for information we haven't used yet. We haven't used the perimeter. That involves the sum of sides! Let's try it. We have $AB + CD + CB + AD = 640$ and $AB = CD = 180$, so $CB + AD = 640 - 180 - 180 = 280$, and we have

$$\cos A = \frac{CB + AD}{2(180)} = \frac{280}{2(180)} = \frac{280}{360} = \frac{7}{9}.$$

□

Exercises

4.2.1 Complete our proof of the Law of Cosines by proving it for obtuse and right triangles.

4.2.2 Two airplanes take off from the same airport at the same time. One flies due west at 200 miles per hour. The other flies 40° north of due east at 250 miles per hour. To the nearest mile, how many miles apart are the planes 90 minutes after takeoff?

4.2.3 Prove each of the following:

(a) If $AB^2 = BC^2 + CA^2$, then $\angle ACB$ is right.

(b) If $AB^2 > BC^2 + CA^2$, then $\angle ACB$ is obtuse.

(c) If $AB^2 < BC^2 + CA^2$, then $\angle ACB$ is acute.

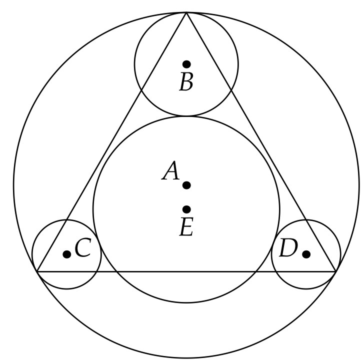
In each case, is the converse of the given statement true? Why or why not? (The **converse** of a statement of the form "If X, then Y" is "If Y, then X.")

4.2.4 One side of a triangle has length twice that of another side, and the third side has length 6. If one angle of the triangle is 120°, then what are the possible values of the lengths of the sides of the triangle?

4.2.5 If $XY = 3$, $YZ = 5$, and $ZX = 7$, then what is $\angle XYZ$?

4.2.6 Quadrilateral $ABCD$ is inscribed in a circle. If $AB = 2$, $BC = 3$, $CD = 4$, and $DA = 6$, then what is AC? *(Source: CEMC)*

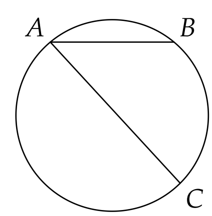
4.2.7★ Equilateral triangle $\mathcal{T}$ is inscribed in circle A, which has radius 10. Circle B with radius 3 is internally tangent to circle A at one vertex of $\mathcal{T}$. Circles C and D, both with radius 2, are internally tangent to circle A at the other two vertices of $\mathcal{T}$. Circles B, C, and D are all externally tangent to circle E. Find the radius of circle E. *(Source: AIME)*



4.2.8★ Triangle ABC has a right angle at B and contains a point P for which $PA = 10$, $PB = 6$, and $\angle APB = \angle BPC = \angle CPA$. Find PC. *(Source: AIME)* **Hints:** 205

4.3 Law of Sines

We start with an important fact from elementary geometry that will be helpful in this section. In the diagram at right, we say that $\angle BAC$ is **inscribed** in arc $\overset{\frown}{BC}$ of the circle because A, B, and C are all on the circle. The measure of an inscribed angle equals half the measure of the arc it intercepts, so $\angle A = \frac{\overset{\frown}{BC}}{2}$. If you are not familiar with this relationship, try proving it yourself before looking it up in a geometry text or online. (This relationship, and many others involving angles and circles, is covered in Art of Problem Solving's *Introduction to Geometry*.)



Problems

Problem 4.14: In $\triangle PQR$, we have $PR = 12$, $\angle QPR = 66°$, and $\angle PRQ = 63°$. In this problem, we find PQ and QR.

(a) Let T be the foot of the altitude from P to $\overline{QR}$. Find PT to the nearest hundredth.

(b) Use PT to find PQ to the nearest hundredth.

(c) Find QR to the nearest hundredth.