

*I've learned that something constructive comes from every defeat. – Tom Landry*

# CHAPTER 4

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## Constructive Counting and 1-1 Correspondences

### 4.1 Introduction

You may recall the concept of **constructive counting** from the book *Introduction to Counting & Probability*. The basic idea is that in order to count the number of items in a certain set, we think about how we would construct an item belonging to that set. During the construction, we keep track of the number of choices that we have at each step. Although this sounds simple, it is a very powerful way to count! In fact, many of the counting problems that we've solved up to now have essentially used this idea.

This chapter consists of constructive counting problems: we're trying to count the number of elements of some set, and we do so by thinking about how we can construct the elements of the set. We've seen lots of problems of this type before, especially in the *Introduction to Counting & Probability* book, but the problems in this section will generally be harder than the ones you've seen before.

We'll also introduce a new tool that's closely related to constructive counting, called a **1-1 correspondence** (read as "one-to-one correspondence"). Here's the basic idea: we want to count the items in some set  $A$ . For whatever reason, set  $A$  is hard to count. But suppose that we can find another set  $B$  such that:

- $B$  has the same number of elements as  $A$ , and
- $B$  is easy to count.

Then to count  $A$ , all we have to do is count  $B$ , and we have our answer! Simple as that!

A "1-1 correspondence" is the tool that we use to do the first part of the above: finding a set  $B$  that has the same number of elements as  $A$ . We do this by showing that every element in  $A$  somehow "matches" a *corresponding* element in  $B$ , and vice versa. Since the elements of  $A$  and  $B$  come in matching pairs, we know that the sets have the same number of elements.

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We can think of using a 1-1 correspondence as a more general version of one of our basic problem-solving strategies: if we don't know how to solve a problem, try to find a simpler, related problem that we *do* know how to solve. That's really what a 1-1 correspondence allows us to do: if we have a set that's hard to count, try to find a related set that's easy to count, and count the easy set instead! As you might expect, the tricky part is finding the "easy" set and showing that it's the same size as the "hard" set.

### 4.2 Some Basic Problems



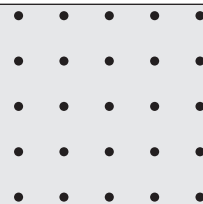
#### Problems

**Problem 4.1:** How many license plates consist of 1 number followed by 3 letters followed by 3 numbers?

**Problem 4.2:** How many 5-digit palindromes are there? (A **palindrome** is a number that reads the same way forwards and backwards. For example, 27872 and 48484 are palindromes, but 28389 and 12541 are not.)

**Problem 4.3:** How many 7-digit numbers have no two adjacent digits equal?

**Problem 4.4:** Four points are chosen at random from the grid at right. What is the probability that the four points are the vertices of a rectangle whose sides are parallel to the sides of the grid?



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The general idea with constructive counting is that we build (or "construct") the items that we're trying to count, and while doing so, keep track of the number of choices that we have at each step in the construction.

The problems in this section are all relatively basic examples of constructive counting, and should be review for you if you've mastered the *Introduction to Counting & Probability* textbook.

**Problem 4.1:** How many license plates consist of 1 number followed by 3 letters followed by 3 numbers?

*Solution for Problem 4.1:* We have 10 choices for the first number, then 26 choices for each of the 3 letters, then 10 choices for each of the last three numbers, for a total of  $10 \cdot 26^3 \cdot 10^3 = 10^4 \cdot 26^3 = 175,760,000$  possible license plates.  $\square$

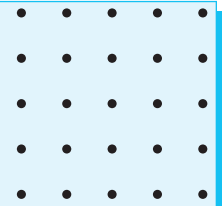
**Problem 4.2:** How many 5-digit palindromes are there? (A **palindrome** is a number that reads the same way forwards and backwards. For example, 27872 and 48484 are palindromes, but 28389 and 12541 are not.)

*Solution for Problem 4.2:* We construct the number from left-to-right. We have 9 choices for the first digit (since it can't be 0), then 10 choices for the second digit, then 10 choices for the third digit. But now we're out of choices—the fourth digit must match the second, and the last digit must match the first. Therefore, there are  $9 \cdot 10 \cdot 10 = 900$  such numbers.  $\square$

**Problem 4.3:** How many 7-digit numbers have no two adjacent digits equal?

*Solution for Problem 4.3:* As in Problem 4.2, we construct these numbers from left-to-right. We have 9 choices for the first digit (since it cannot be 0). Then, no matter what we choose for the first digit, we have 9 choices for the second digit—any digit except the one that we chose for the first digit. Similarly, for each subsequent digit, we have 9 choices, since each digit can be any of 0 through 9 as long as it does not match the previous digit. Therefore, there are 9 choices for every digit, and hence there are  $9^7 = 4,782,969$  such numbers.  $\square$

**Problem 4.4:** Four points are chosen at random from the grid at right. What is the probability that the four points are the vertices of a rectangle whose sides are parallel to the sides of the grid?



*Solution for Problem 4.4:* There are  $\binom{25}{4}$  ways in which we can choose 4 vertices (without regard to order). We need to determine how many of these sets of 4 vertices produce a valid rectangle. How can we construct such a rectangle? There are a couple of different ways we could proceed.

*Solution 1:* A rectangle whose sides are parallel to the sides of the grid is determined by its two opposite corners. There are  $\binom{25}{2} = 300$  pairs of points in the grid, but there are two things we have to worry about. First, if we pick two points in the same row or same column, then we won't get a rectangle. There are 5 rows and 5 columns, and each has  $\binom{5}{2} = 10$  pairs of points, so we have to subtract  $(10)(10) = 100$  from our count, leaving  $300 - 100 = 200$  pairs of points left. Second, each rectangle has two pairs of opposite corners, so we've overcounted by a factor of 2, and hence our final count is  $200/2 = 100$ .

*Solution 2:* A rectangle whose sides are parallel to the sides of the grid can be determined by choosing two rows to form the horizontal sides and choosing two columns to form the vertical sides. We can choose two rows in  $\binom{5}{2} = 10$  ways and choose two columns in  $\binom{5}{2} = 10$  ways, so the number of rectangles is  $(10)(10) = 100$ .

Therefore, 100 of the  $\binom{25}{4}$  sets of 4 vertices form valid rectangles, and hence our probability is  $\frac{100}{\binom{25}{4}} = \frac{2}{253}$ .

We could also constructively count the probability directly, without counting the number of rectangles. For this, we think about selecting the vertices in order, one at a time. The first vertex is arbitrarily chosen. For the second vertex, there are two exclusive possibilities:

*Case 1:* The second vertex is in the same row or column as the first vertex. This occurs with probability  $\frac{8}{24} = \frac{1}{3}$ . Then, in order to produce a valid rectangle, the third vertex must be on one of the lines containing one of the first two vertices and perpendicular to the line containing the first two vertices. There are 8 such points on these lines, and 23 points remaining to choose from, so the probability of choosing one is  $\frac{8}{23}$ . Finally, only one point of the 22 remaining will complete the rectangle, and the probability of

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choosing it is  $\frac{1}{22}$ .

*Case 2:* The second vertex is not in the same row or column as the first vertex. This occurs with probability  $\frac{16}{24} = \frac{2}{3}$ . These two points must be opposite corners of our rectangle, so the final two points that we choose must be the other two corners. They are chosen with probability  $\frac{2}{23} \cdot \frac{1}{22}$ .

Therefore, the probability of choosing the four corners of a valid rectangle is:

$$\frac{1}{3} \cdot \frac{8}{23} \cdot \frac{1}{22} + \frac{2}{3} \cdot \frac{2}{23} \cdot \frac{1}{22} = \frac{12}{3 \cdot 23 \cdot 22} = \frac{2}{253}.$$

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### Exercises

**4.2.1** A dot is marked at each vertex of a triangle  $ABC$ . Then 2, 3, and 7 more dots are marked on the sides  $\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{CA}$ , respectively. How many triangles have their vertices at these dots? (*Source: HMMT*)

**4.2.2** Consider the set  $S = \{1, 2, 3, \dots, 34\}$ . How many ways are there to choose (without regard to order) three numbers from  $S$  whose sum is divisible by 3? (*Source: ARML*)

**4.2.3** How many orderings of the letters in MISSISSIPPI read the same forwards as backwards?

**4.2.4** Nine tiles are numbered  $1, 2, 3, \dots, 9$ . Each of three players randomly selects and keeps three of the tiles, and sums those three values. Find the probability that all three players obtain an odd sum. (*Source: AIME*) **Hints:** 131

**4.2.5★** Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins. (*Source: AIME*) **Hints:** 13

## 4.3 Harder Constructive Counting Problems

### Problems

**Problem 4.5:** We wish to compute the sum of all of the 5-digit palindromes.

- (a) How many 5-digit palindromes are there?
- (b) How many have '1' as the last digit? How many have '2' as the last digit? And so on?
- (c) What is the units digit of the sum of all of the 5-digit palindromes?
- (d) Can you extend your reasoning from parts (a)-(c) above to find the sum of all of the 5-digit palindromes?