6.3.2 Check the two divisions we performed in Problem 6.12 by multiplying the quotient by the divisor, then adding the remainder.

6.3.3 Find the quotient and remainder when $x^5 - 23x^3 + 11x^2 - 14x + 20$ is divided by $x + 5$.

6.3.4 Find the quotient and remainder when $4x^4 - 10x^3 + 14x^2 + 7x - 19$ is divided by $2x - 1$.

6.3.5 Find the quotient and remainder when $3z^3 - 4z^2 - 14z + 3$ is divided by $3z + 5$.

6.3.6 Find the quotient and remainder when $x^4 + 3x^3 - x^2 + 7x - 1$ is divided by $2 - x$.

6.3.7 Stan divided $x^3 + 3x^2 - 7x + 5$ by $2x - 3$ using synthetic division. He found a quotient of $x^2 + 6x + 11$ and a remainder of 38. By looking at his answer, he immediately realized that he had made a mistake. How could he tell? What mistake did he probably make?

6.4 The Remainder Theorem

Problems

Problem 6.13: Let $f(x) = x^4 - 3x^3 + 7x^2 - x + 5$.

(a) What is the remainder when $f(x)$ is divided by $x - 3$?

(b) What is $f(3)$?

Problem 6.14: In this problem we prove the **Remainder Theorem** for polynomials. Suppose $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when the polynomial $f(x)$ is divided by $x - a$, where a is a constant.

(a) What is the degree of $r(x)$?

(b) Write an equation expressing $f(x)$ in terms of $q(x)$, $r(x)$, and $x - a$.

(c) Use part (b) to show that $f(a) = r(a)$.

(d) Why do (a) and (c) together tell us that the remainder when the polynomial $f(x)$ is divided by $x - a$ is $f(a)$?

Problem 6.15: Let $f(x) = x^4 - 6x^3 - 9x^2 + 20x + 7$. Use synthetic division to find $f(7)$.

Problem 6.16: Find the remainder when $2x^{10} - 3ix^8 + (1 + i)x^2 - (3 + 2i)x + 1$ is divided by $x - i$.

Problem 6.17: Let $f(x) = x^{10} - 2x^5 + 3$.

(a) Find the remainder when $f(x)$ is divided by $x - 2$.

(b) Find the remainder when $f(x)$ is divided by $2x - 4$.

Extra! *Imagination is the beginning of creation. You imagine what you desire, you will what you imagine and at last you create what you will.*

– George Bernard Shaw

Problem 6.18: Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x - 19$, the remainder is 99, and when $P(x)$ is divided by $x - 99$, the remainder is 19. In this problem we find the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$. *(Source: AHSME)*

(a) Find $P(19)$ and $P(99)$.

(b) Let $r(x)$ be the desired remainder. Write an equation representing the division of $P(x)$ by $(x - 19)(x - 99)$.

(c) What is the degree of $r(x)$?

(d) Choose two convenient values of x and substitute these into your equation from (b). Use the results to find $r(x)$.

Problem 6.19: Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$. *(Source: Great Britain)*

Problem 6.13: Let $f(x) = x^4 - 3x^3 + 7x^2 - x + 5$.

(a) What is the remainder when $f(x)$ is divided by $x - 3$?

(b) What is $f(3)$?

Solution for Problem 6.13:

$$\begin{array}{r|rrrrr} 3 & 1 & -3 & 7 & -1 & 5 \\ & & 3 & 0 & 21 & 60 \\ \hline & 1 & 0 & 7 & 20 & \multicolumn{1}{|r}{65} \end{array}$$

(a) We find the remainder using the synthetic division shown at right. The remainder is 65.

(b) We have $f(3) = 3^4 - 3 \cdot 3^3 + 7 \cdot 3^2 - 3 + 5 = 65$.

□

Hmm.... We see that $f(3)$ equals the remainder when $f(x)$ is divided by $x - 3$. Is this a coincidence?

Problem 6.14: Suppose $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when the polynomial $f(x)$ is divided by $x - a$, where a is a constant. Is it always true that the remainder equals $f(a)$?

Solution for Problem 6.14: Because $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when the polynomial $f(x)$ is divided by $x - a$, we have

$$f(x) = (x - a)q(x) + r(x).$$

Because either the remainder is 0 or its degree is less than the degree of the divisor $x - a$, we see that $r(x)$ must be a constant. So, we let $r(x) = c$ for some constant c, and we have

$$f(x) = (x - a)q(x) + c.$$

This equation must hold for all x, and we're interested in $f(a)$, so we let $x = a$ in the equation. This gives

$$f(a) = (a - a)q(a) + c = 0 \cdot q(a) + c = c.$$

Therefore, when a polynomial $f(x)$ is divided by $x - a$, the remainder is $f(a)$. □

Important: The **Remainder Theorem** states that when a polynomial $f(x)$ is divided by $x-a$, the remainder is $f(a)$.

Problem 6.15: Let $f(x) = x^4 - 6x^3 - 9x^2 + 20x + 7$. Use synthetic division to find $f(7)$.

Solution for Problem 6.15: We can put $x = 7$ into $f(x)$ to evaluate $f(7)$, but the Remainder Theorem allows us to evaluate $f(7)$ with less complicated arithmetic. The synthetic division at right tells us that the remainder when $f(x)$ is divided by $x-7$ is 49. Therefore, we have $f(7) = 49$. □

7	1	−6	−9	20	7
		7	7	−14	42
	1	1	−2	6	49

The Remainder Theorem can also be used to find remainders without division.

Problem 6.16: Find the remainder when $2x^{10} - 3ix^8 + (1+i)x^2 - (3+2i)x + 1$ is divided by $x - i$.

Solution for Problem 6.16: We could use synthetic division, even though some of our coefficients are nonreal. However, because we're dividing a 10^{th} degree polynomial, this synthetic division would be pretty lengthy. Instead, we use the Remainder Theorem. We let $f(x) = 2x^{10} - 3ix^8 + (1+i)x^2 - (3+2i)x + 1$. By the Remainder Theorem, the remainder when we divide $f(x)$ by $x - i$ is

$$\begin{aligned} f(i) &= 2 \cdot i^{10} - 3i \cdot i^8 + (1+i) \cdot i^2 - (3+2i) \cdot i + 1 \\ &= 2(-1) - 3i(1) + (1+i)(-1) - 3i - 2i^2 + 1 \\ &= -7i. \end{aligned}$$

□

Problem 6.17: Let $f(x) = x^{10} - 2x^5 + 3$. Find the remainder when $f(x)$ is divided by $2x - 4$.

Solution for Problem 6.17: What's wrong with this solution:

Bogus Solution: The remainder is $f(4) = 4^{10} - 2 \cdot 4^5 + 3 = 1046531$.

The Remainder Theorem tells us that when $f(x)$ is divided by $x - a$, the remainder is $f(a)$. The value of $f(4)$ equals the remainder when $f(x)$ is divided by $x - 4$, not $2x - 4$.

WARNING!! The Remainder Theorem applies to division by linear expressions with leading coefficient 1.

While we can't directly apply the Remainder Theorem, we can use our proof of the Remainder Theorem as guidance. Because we are dividing by a linear expression, the remainder must be a constant. So, we seek the constant r such that

$$f(x) = (2x-4)q(x) + r$$

for all values of x. When proving the Remainder Theorem, we eliminated $q(x)$ by setting x equal to a value that made the divisor 0. Here, we let $x = 2$ to find $f(2) = (2 \cdot 2 - 4)q(2) + r = r$. So, our remainder is $f(2) = 2^{10} - 2 \cdot 2^5 + 3 = 963$. □

Notice that the key step in this solution is not to apply the Remainder Theorem directly, but rather to use the same general technique we used to prove the Remainder Theorem.

Concept: Don't just memorize important theorems. Learn how to prove them. The methods used to prove important theorems can also be used to solve other problems to which it is not easy to directly apply these theorems.

Here, the technique we use is:

Concept: Suppose $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when the polynomial $f(x)$ is divided by the polynomial $d(x)$. One powerful way to learn more about $r(x)$ is to express the division as

$$f(x) = q(x) \cdot d(x) + r(x),$$

and then eliminate the $q(x) \cdot d(x)$ term by choosing values of x such that $d(x) = 0$. Combining this with the fact that $\deg r$ is no greater than $\deg d$ will often allow us to determine $r(x)$.

This strategy works even when $d(x)$ is not linear.

Problem 6.18: Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x - 19$, the remainder is 99, and when $P(x)$ is divided by $x - 99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$? *(Source: AHSME)*

Solution for Problem 6.18: Because we are dividing by a quadratic, the degree of the remainder is no greater than 1. So, the remainder is $ax + b$, for some constants a and b. Therefore, we have

$$P(x) = (x - 19)(x - 99)Q(x) + ax + b,$$

where $Q(x)$ is the quotient when $P(x)$ is divided by $(x - 19)(x - 99)$. We can eliminate the $Q(x)$ term by letting $x = 19$ or by letting $x = 99$. Doing each in turn gives us the system of equations

$$\begin{aligned} P(19) &= 19a + b = 99, \\ P(99) &= 99a + b = 19. \end{aligned}$$

Solving this system of equations gives us $a = -1$ and $b = 118$. So, the remainder is $-x + 118$. □

Problem 6.19: Find the remainder when the polynomial $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$. *(Source: Great Britain)*

Solution for Problem 6.19: Let $P(x) = x^{81} + x^{49} + x^{25} + x^9 + x$ and let $Q(x)$ be the quotient when $P(x)$ is divided by $x^3 - x$. We are dividing by a cubic, so the degree of the remainder is no more than 2.

Therefore, the remainder is $ax^2 + bx + c$ for some constants a, b, and c. So, we have

$$P(x) = (x^3 - x)Q(x) + ax^2 + bx + c.$$

Since $x^3 - x = x(x+1)(x-1)$, we can let $x = -1, 0$, or 1 to make $x^3 - x$ equal to 0. Doing each in turn gives us the system of equations

$$\begin{aligned} P(-1) &= a - b + c, \\ P(0) &= c, \\ P(1) &= a + b + c. \end{aligned}$$

Since $P(0) = 0$, we have $c = 0$. Adding the first and third equations then gives $2a = P(-1) + P(1) = -5 + 5 = 0$, so $a = 0$. Finally, from $P(1) = a + b + c$, we have $b = P(1) = 5$, so the desired remainder is $5x$. □

Exercises

6.4.1 Let $g(x) = 2x^6 - x^4 + 4x^2 - 8$. Find the remainder when $g(x)$ is divided by each of the following:

(a) $x + 1$

(b) $x - 1$

(c) $x + 3$

(d) $x - 3$

(e) $x^2 + 4x + 3$

(f) $x^2 - 4x + 3$

6.4.2 When $y^2 + my + 2$ is divided by $y - 1$, the quotient is $f(y)$ and the remainder is R_1. When $y^2 + my + 2$ is divided by $y + 1$, the quotient is $g(y)$ and the remainder is R_2. If $R_1 = R_2$, then find m. *(Source: AHSME)*

6.4.3 Find the remainder when x^{100} is divided by $(x - 1)(x - 2)$. *(Source: AHSME)* (You can leave constants in the form a^b in your answer.)

6.4.4 Suppose $q(x)$ and $r(x)$ are the quotient and remainder, respectively, when the polynomial $f(x)$ is divided by the polynomial $d(x)$. Show that if $x = a$ is a root of $d(x)$, then $r(a) = f(a)$.

6.4.5★ Find the remainder when $x^{100} - 4x^{98} + 5x + 6$ is divided by $x^3 - 2x^2 - x + 2$. **Hints:** 91

6.5 Summary

A **polynomial** in one variable is an expression or function of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$

where x is a variable and each a_i is a constant. Each individual summand $a_i x^i$ is called a **term**, and each number a_i is the **coefficient** of the corresponding x^i. The **degree** of a polynomial is the greatest integer n such that the coefficient a_n is not equal to 0. (The degree of the polynomial 0 is undefined.)

Important: When the polynomial $f(x)$ is divided by the polynomial $d(x)$, the quotient $q(x)$ and remainder $r(x)$ are the polynomials such that

$$f(x) = q(x) \cdot d(x) + r(x)$$

and either $r(x) = 0$ or $\deg r < \deg d$.