(a) Consider the circle with diameter $\overline{OP}$. Call this circle C. Why must C hit $\odot O$ in at least two different points?

(b) Why is it impossible for C to hit $\odot O$ in *three* different points? **Hints:** 530

(c) Let the points where C hits $\odot O$ be A and B. Prove that $\angle PAO = \angle PBO = 90^\circ$.

(d) Prove that $\overline{PA}$ and $\overline{PB}$ are tangent to $\odot O$.

(e)★ Now for the tricky part – proving that these are the only two tangents. Suppose D is on $\odot O$ such that $\overline{PD}$ is tangent to $\odot O$. Why must D be on C? **Hints:** 478

(f) Why does the previous part tell us that $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are the only lines through P tangent to $\odot O$?

12.3.6★ $\odot O$ is tangent to all four sides of rhombus $ABCD$, $AC = 24$, and $AB = 15$.

(a) Prove that $\overline{AC}$ and $\overline{BD}$ meet at O (i.e., prove that the intersection of the diagonals of $ABCD$ is the center of the circle.) **Hints:** 439

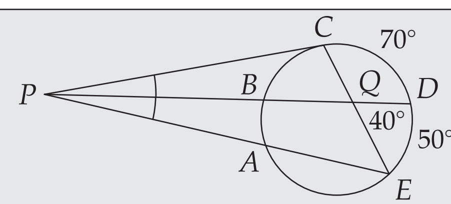
(b) What is the area of $\odot O$? **Hints:** 508

12.4 Problems

Problems

Problem 12.19: A quadrilateral is said to be a **cyclic quadrilateral** if a circle can be drawn that passes through all four of its vertices. Prove that if $ABCD$ is a cyclic quadrilateral, then $\angle A + \angle C = 180^\circ$.

Problem 12.20: In the figure, $\overline{PC}$ is tangent to the circle and $\overline{PD}$ bisects $\angle CPE$. Furthermore, $\widehat{CD} = 70^\circ$, $\widehat{DE} = 50^\circ$, and $\angle DQE = 40^\circ$. In this problem we determine the measure of the arc from A to E that does not include point C.



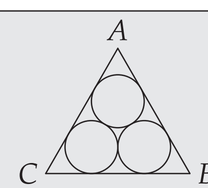
(a) Find $\widehat{BC}$.

(b) Find $\angle CPB$.

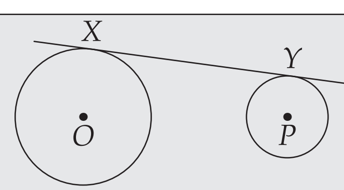
(c) Find $\widehat{AB}$.

(d) Finish the problem.

Problem 12.21: Three congruent circles with radius 1 are drawn inside equilateral $\triangle ABC$ such that each circle is tangent to the other two and to two sides of the triangle. Find the length of a side of $\triangle ABC$. **Hints:** 428

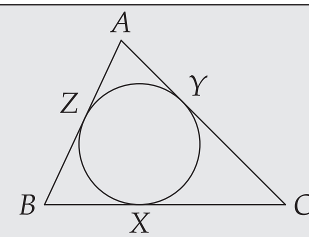


Problem 12.22: $\overleftrightarrow{XY}$ is tangent to both circle O and circle P. Given that $OP = 40$, and the radii of circles O and P are 31 and 7, respectively, find XY. **Hints:** 548, 267



Problem 12.23: Median $\overline{AM}$ of $\triangle ABC$ has length 8. Given that $BC = 16$ and $AB = 9$, find the area of $\triangle ABC$. **Hints:** 25, 187

Problem 12.24: The diagram shows the incircle of $\triangle ABC$. X, Y, and Z are the points of tangency where the incircle touches the triangle. In this problem we will find an expression for AZ in terms of the sides of the triangle.



(a) Find equal segments in the diagram and assign them variables.

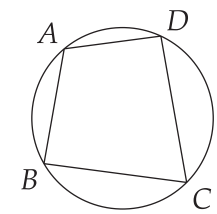
(b) Let $AB = c$, $AC = b$, and $BC = a$. Use your variables from the first part to write equations that include these lengths.

(c) Solve your resulting equations for AZ. **Hints:** 157

We'll now apply our knowledge of circles, angles, and tangents to more challenging problems and develop some useful geometric concepts.

Problem 12.19: A quadrilateral is said to be a **cyclic quadrilateral** if a circle can be drawn that passes through all four of its vertices. Prove that if $ABCD$ is a cyclic quadrilateral, then $\angle A + \angle C = 180^\circ$.

Solution for Problem 12.19: Since $\angle A$ and $\angle C$ are inscribed angles, we have



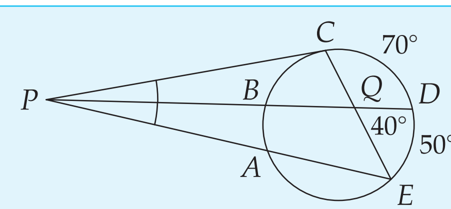
$$\angle A + \angle C = \frac{\widehat{BCD}}{2} + \frac{\widehat{BAD}}{2} = \frac{\widehat{BCD} + \widehat{BAD}}{2} = \frac{360^\circ}{2} = 180^\circ.$$

□

Important: A quadrilateral is a **cyclic quadrilateral** if a circle can be drawn that passes through all four of its vertices. Such a quadrilateral is said to be **inscribed** in the circle. The opposite angles of any cyclic quadrilateral sum to 180°.

You'll be seeing a lot more of cyclic quadrilaterals when you move into more advanced geometry.

Problem 12.20: In the figure, $\overline{PC}$ is tangent to the circle and $\overline{PD}$ bisects $\angle CPE$. If $\widehat{CD} = 70^\circ$, $\widehat{DE} = 50^\circ$, and $\angle DQE = 40^\circ$, then determine the measure of the arc from A to E that does not include point C.



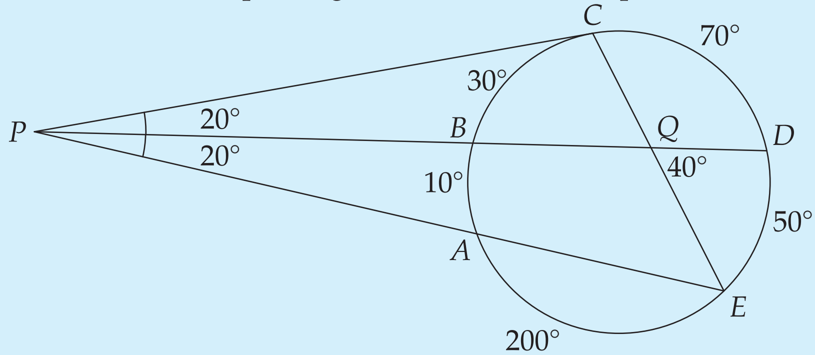
Solution for Problem 12.20: We can't directly find the desired arc, so we try finding whatever we can. First, we note that since $\angle DQE$ is the average of $\widehat{DE}$ and $\widehat{CB}$, we have $\widehat{CB} = 30^\circ$. Now that we have $\widehat{CB}$, we can find $\angle CPD$:

$$\angle CPD = (\widehat{CD} - \widehat{CB})/2 = 20^\circ.$$

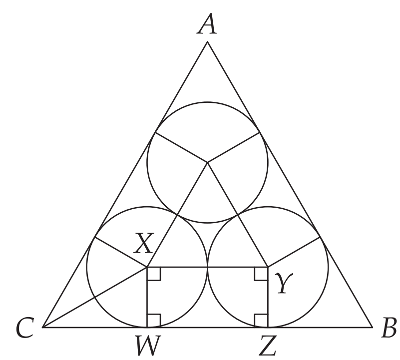
Since $\overline{DP}$ bisects $\angle CPE$, we know $\angle DPE = \angle CPD = 20^\circ$. Because $\angle DPE = (\widehat{DE} - \widehat{AB})/2$, we find $\widehat{AB} = 10^\circ$.

Now that we have all the other arcs of the circle, we can find our desired arc by subtracting from 360°: $360° - 50° - 70° - 30° - 10° = 200°$. □

Concept: Label all the given information in the problem, and all the information you find as you find it. This will help you discover new facts about the problem. For example, your final diagram in the last problem might look like the figure below. Notice that our various arc measures and angles are marked, and that the equal angles at P are marked equal.



Problem 12.21: Three congruent circles with radius 1 are drawn inside equilateral $\triangle ABC$ such that each circle is tangent to the other two and to two sides of the triangle. Find the length of a side of $\triangle ABC$.

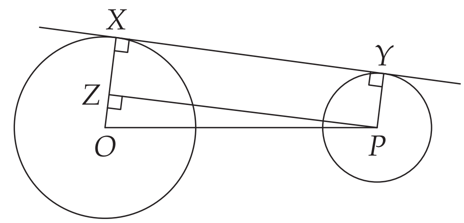


Solution for Problem 12.21: We need to create simple figures to work with, so we start by connecting the centers of our circles and drawing radii to tangent points. (Note that connecting the centers of the circles is the same as drawing radii to where the circles are tangent.) Since $WXYZ$ is a rectangle (because $XW = YZ$, $\overline{XW} \parallel \overline{YZ}$, and $\angle XWZ = 90°$), we have $WZ = XY = 2$. Hence, we need only find CW to finish, since BZ is the same as CW.

We draw $\overline{CX}$ to build a right triangle and note that this segment bisects $\angle ACB$ because circle X is tangent to both $\overline{AC}$ and $\overline{BC}$ (and hence its center is equidistant from them). Since $\angle XCW = (\angle ACB)/2 = 30°$, $\triangle CXW$ is a 30-60-90 triangle. Thus, $CW = XW\sqrt{3} = \sqrt{3}$. Finally, we have $BC = BZ + ZW + WC = 2 + 2\sqrt{3}$. □

Concept: When you have tangents in a problem, it's often very helpful to draw radii to points of tangency to build right triangles. When you have tangent circles, connect the centers. (In fact, if you have multiple circles in a problem, connecting the centers will sometimes help even when the circles aren't tangent.)

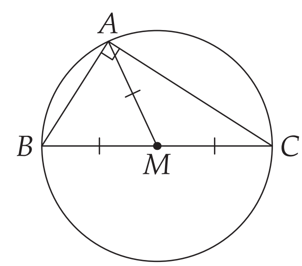
Problem 12.22: X is on circle O and Y on circle P such that $\overleftrightarrow{XY}$ is tangent to both circles. Given that $OP = 40$, and the radii of circles O and P are 31 and 7, respectively, find XY.



Solution for Problem 12.22: We start with the usual segments to draw: $\overline{XY}$, the radii to points of tangency, and the segment connecting the centers. We still don't have a right triangle to work with, but we do know that radii $\overline{OX}$ and $\overline{PY}$ are both perpendicular to tangent $\overline{XY}$ as shown. Since both $\overline{OX}$ and $\overline{PY}$ are perpendicular to the same line, they are parallel. We make a right triangle and a rectangle by drawing a line through P parallel to $\overline{XY}$. $ZX = PY = 7$, so $OZ = OX - ZX = 24$. Since $OP = 40$, we have $PZ = 32$ from right triangle $\triangle OZP$. Since $XYPZ$ is a rectangle, we have $XY = ZP = 32$, so the length of the common tangent is 32.

$\overline{XY}$ is called a common *external* tangent of the two circles. As an Exercise, you'll find the length of the common *internal* tangent, too. □

Problem 12.23: Median $\overline{AM}$ of $\triangle ABC$ has length 8. Given that $BC = 16$ and $AB = 9$, find the area of $\triangle ABC$.



Solution for Problem 12.23: When we draw the figure and label all our lengths, we see that $BM = AM = CM = 8$. Therefore, a circle centered at M with radius 8 goes through all three vertices of $\triangle ABC$. Since $\overline{BC}$ is a diameter of this circle, $\angle BAC$ is inscribed in a semicircle and therefore must be a right angle. So, $AC = \sqrt{BC^2 - AB^2} = 5\sqrt{7}$. $\triangle ABC$ is a right triangle, so its area is half the product of its legs:

$$[ABC] = \frac{(AB)(AC)}{2} = \frac{(9)(5\sqrt{7})}{2} = \frac{45\sqrt{7}}{2}.$$

□

Using this same reasoning, we can also prove this important fact:

Important: If the length of a median of a triangle is half the length of the side to which it is drawn, the triangle must be a right triangle. Moreover, the side to which this median is drawn is the hypotenuse of the right triangle.

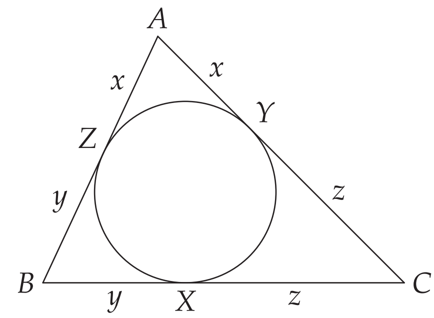
We can also look to this problem for some important problem solving techniques:

Concept: When stuck on a problem, always ask yourself 'Where have I seen something like this before?' In Problem 12.23, we have a median that is half the side to which it is drawn. This should make us think of right triangles, since the median to the hypotenuse of a right triangle is half the hypotenuse. Then we go looking for right triangles.

Concept: Always be on the lookout for right triangles.

Problem 12.24: Let Z be the point where the incircle of $\triangle ABC$ meets $\overline{AB}$. Find AZ in terms of the sides of $\triangle ABC$.

Solution for Problem 12.24: We start by labeling the equal tangents from the vertices as shown in the diagram. We want to relate x to the sides of the triangle, so we write the sides of the triangle in terms of x, y, and z. We let the sides of the triangle be $AB = c$, $AC = b$, and $BC = a$ and we have:



$$\begin{aligned} AB &= c = x + y \\ AC &= b = x + z \\ BC &= a = y + z \end{aligned}$$

We want x in terms of a, b, and c. Adding the three equations will give us $x + y + z$, which we can use with $y + z = a$ to find x:

$$a + b + c = 2(x + y + z) \qquad \text{so} \qquad x + y + z = \frac{a+b+c}{2}.$$

We can then subtract the equation $y + z = a$ from $x + y + z = (a + b + c)/2$ to find

$$x = \frac{a+b+c}{2} - a = s - a,$$

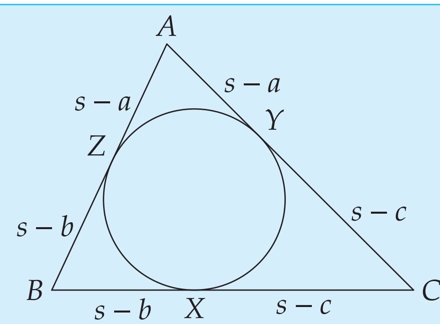
where s is the semiperimeter (half the perimeter) of the triangle. Similarly, $y = s - b$ and $z = s - c$. □

Concept: Symmetric systems of equations can often be easily solved by either multiplying all the equations or adding them.

Important: The lengths from the vertices of $\triangle ABC$ to the points of tangency of its incircle are given as follows:

$$\begin{aligned} AZ &= AY = s - a \\ BZ &= BX = s - b \\ CX &= CY = s - c \end{aligned}$$

where $AB = c$, $AC = b$, and $BC = a$, and the semiperimeter of $\triangle ABC$ is s.



Exercises

Problems 12.4.3, 12.4.4, 12.4.5, and 12.4.8 are very important relationships that you'll be seeing again in your study of more advanced geometry. Be sure to pay special attention to them.

12.4.1 Is every quadrilateral cyclic?

12.4.2 Prove the following about cyclic quadrilaterals:

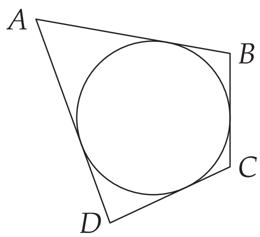
(a) A cyclic parallelogram must be a rectangle.

(b) A cyclic rhombus must be a square.

(c) A cyclic trapezoid must be isosceles.

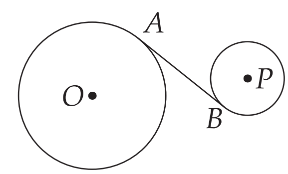
12.4.3 Prove that if median $\overline{XM}$ of $\triangle XYZ$ has half the length of $\overline{YZ}$, then $\triangle XYZ$ is a right triangle with hypotenuse $\overline{YZ}$.

12.4.4 $ABCD$ is a cyclic quadrilateral. Prove that $\angle ACB = \angle ADB$.



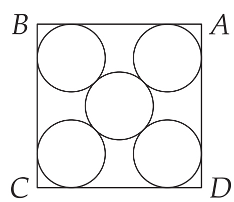
12.4.5 Quadrilateral $ABCD$ in the diagram at left is a **circumscribed quadrilateral**, meaning that it is circumscribed about a circle, so the circle is tangent to all four sides of $ABCD$. Show that $AB + CD = BC + AD$. **Hints:** 570

12.4.6 Does every quadrilateral have an inscribed circle (a circle tangent to all four sides), as $ABCD$ does in the previous problem?



12.4.7 In the figure at right, $\overline{AB}$ is tangent to both $\odot O$ and $\odot P$. The radius of $\odot O$ is 8, the radius of $\odot P$ is 4, and $OP = 36$. find AB. (A common tangent like $\overline{AB}$ is sometimes called the **common internal tangent** of two circles.) **Hints:** 591

12.4.8 Prove that the inradius of a right triangle with legs of length a and b and hypotenuse c is $(a + b - c)/2$.



12.4.9★ The five circles in the diagram are congruent and $ABCD$ is a square with side length 4. The four outer circles are each tangent to the middle circle and to the square on two sides as shown. Find the radius of each of the circles. **Hints:** 543, 416

12.5 Construction: Tangents

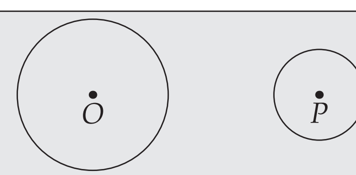
Now you'll use your newfound tangent knowledge to construct tangents to circles.

Problems

Problem 12.25: Given a circle with center O and point A on the circle, construct a line through A that is tangent to the circle.

Problem 12.26: Given a circle with center O and point P outside the circle, construct a line through P that is tangent to the circle.

Problem 12.27: Nonintersecting circles with centers O and P are shown. Construct a line that is tangent to *both* circles.



Extra! *Seek simplicity, and distrust it.*

–Alfred North Whitehead