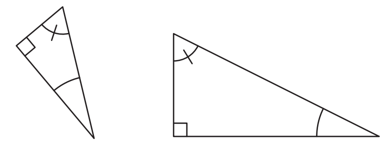
Similar figures do not need to have the same orientation. The diagram to the right shows two similar triangles with different orientations.



Speaking of triangles, we'll be spending the rest of this chapter discussing how to tell when two triangles are similar, and how to use similar triangles once we find them. Below are a couple Exercises that provide practice using triangle similarities to write equations involving side lengths.

Exercises

5.1.1 Given that $\triangle ABC \sim \triangle YXZ$, which of the statements below must be true?

(a) $AB/YX = AC/YZ$.

(b) $AB/BC = YX/XZ$.

(c) $AB/XZ = BC/YX$.

(d) $(AC)(YX) = (YZ)(BA)$.

(e) $BC/BA = XY/ZY$.

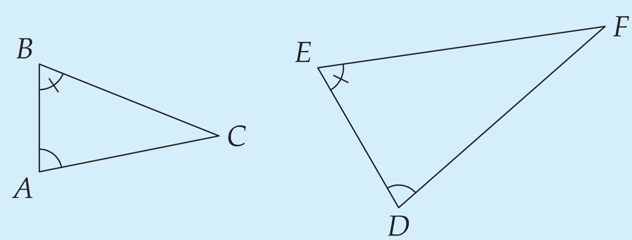
5.1.2 $\triangle ABC \sim \triangle ADB$, $AC = 4$, and $AD = 9$. What is AB? *(Source: MATHCOUNTS)* **Hints:** 113

5.2 AA Similarity

In our introduction, we stated that similar figures have all corresponding angles equal, and that corresponding sides are in a constant ratio. It sounds like a lot of work to prove all of that; however, just as for triangle congruence, we have some shortcuts to prove that triangles are similar. We'll start with the most commonly used method.

Important: **Angle-Angle Similarity** (**AA Similarity**) tells us that if two angles of one triangle equal two angles of another, then the triangles are similar. $\angle A = \angle D$ and $\angle B = \angle E$ together imply $\triangle ABC \sim \triangle DEF$, so

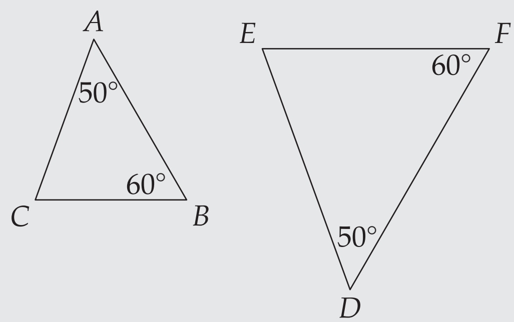
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$



We'll explore why AA Similarity works in Section 5.5, but first we'll get some experience using it in some problems.

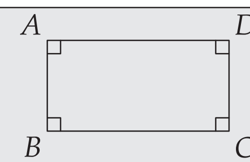
Problems

Problem 5.2: Below are two triangles that have the same measures for two angles.

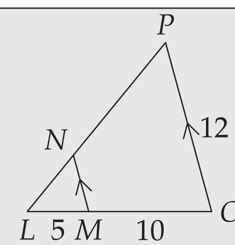


Find the third angle in each, and find the ratios AB/DF, AC/DE, BC/EF by measuring the sides with a ruler.

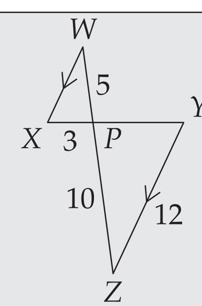
Problem 5.3: In this problem we try to extend AA Similarity to figures with more angles by considering figures with four angles. Can you create a figure $EFGH$ that has the same angles as $ABCD$ at right such that $EFGH$ and $ABCD$ are not similar? (In other words, can you create $EFGH$ so that the angles of $EFGH$ equal those of $ABCD$, but the ratio of corresponding sides between $EFGH$ and $ABCD$ is not the same for all corresponding pairs of sides?)



Problem 5.4: In the figure at right, $\overline{MN} \parallel \overline{OP}$, $OP = 12$, $MO = 10$, and $LM = 5$. Find MN.



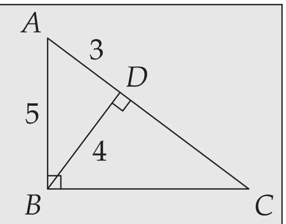
Problem 5.5: The lengths in the diagram are as marked, and $\overline{WX} \parallel \overline{YZ}$. Find PY and WX.



Extra! My dad was going to cut down a dead tree in our yard one day, but he was afraid it might hit some nearby power lines. He knew that if the tree were over 45 feet tall, the tree would hit the power lines. He stood 30 feet from the base of the tree and held a ruler 6 inches in front of his eye.

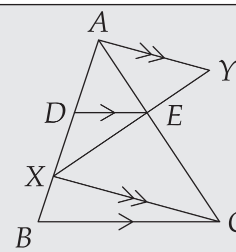
Continued on the next page. . .

Problem 5.6: Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.

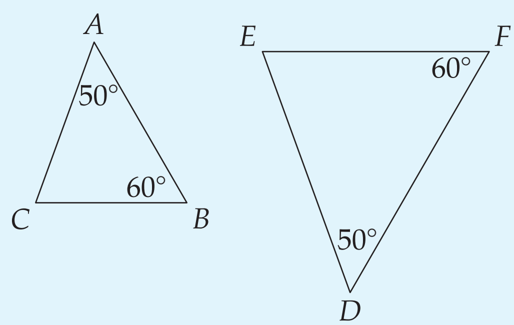


Problem 5.7: Given that $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that

$$\frac{EY}{EX} = \frac{AD}{DB}.$$



Problem 5.2: Below are two triangles that have the same measures for two angles.



Find the third angle in each, and find the ratios AB/DF, AC/DE, BC/EF by measuring the sides with a ruler.

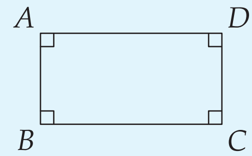
Solution for Problem 5.2: The last angle in each triangle is $180^\circ - 50^\circ - 60^\circ = 70^\circ$, so the angles of $\triangle ABC$ match those of $\triangle DFE$. In the same way, if we ever have two angles of one triangle equal to two angles of another, we know that the third angles in the two triangles are equal.

Measuring, we find that the ratios are each about 2/3. It appears to be the case that if all the angles of two triangles are equal, then the two triangles are similar. □

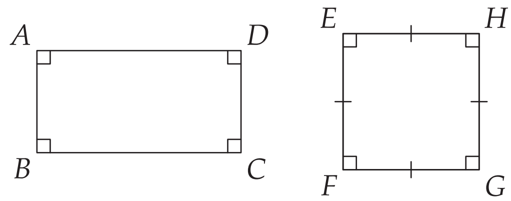
We might wonder if two figures with equal corresponding angles are always similar. So, we add an angle and see if it works for figures with four angles.

Extra! *. . . continued from the previous page*
He lined the bottom of the ruler up with the base of the tree, and saw that the top of the tree lined up with a point 8 inches high on the ruler. He then knew he could safely cut the tree down. How did he know?

Problem 5.3: Does your rule work for figures with more than 3 angles? Can you create a figure $EFGH$ that has the same angles as $ABCD$ at right such that $EFGH$ and $ABCD$ are not similar? (In other words, can you create $EFGH$ so that the angles of $EFGH$ equal those of $ABCD$, but the ratio of corresponding sides between $EFGH$ and $ABCD$ is not the same?)



Solution for Problem 5.3: We can quickly find such an $EFGH$. The diagram to the right shows a square $EFGH$ next to our initial rectangle. Clearly these figures have the same angles, but when we check the ratios, we find that

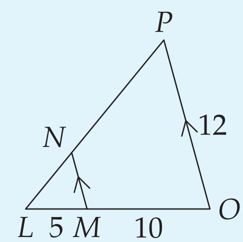


$$\frac{AB}{EF} < 1 < \frac{BC}{FG}.$$

$ABCD$ and $EFGH$ are not similar, so equal angles are not enough to prove similarity here. □

Let's return to triangles and tackle some problems using AA Similarity.

Problem 5.4: In the figure at right, $\overline{MN} \parallel \overline{OP}$, $OP = 12$, $MO = 10$, and $LM = 5$. Find MN.

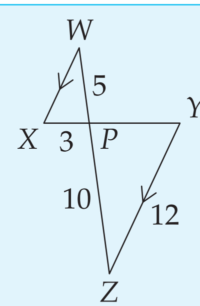


Solution for Problem 5.4: See if you can find the flaw in this solution:

Bogus Solution: Since $\overline{MN} \parallel \overline{OP}$, we have $\angle LMN = \angle LOP$ and $\angle LNM = \angle LPO$. Therefore, $\triangle LMN \sim \triangle LOP$, so $LM/MO = MN/OP$. Substituting our given side lengths gives $5/10 = MN/12$, so $MN = 6$.

Everything in this solution is correct except for $LM/MO = MN/OP$. $\overline{MO}$ is not a side of one of our similar triangles! The correct equation is $LM/LO = MN/OP$. Since $LO = LM + MO = 15$, we now have $5/15 = MN/12$, so $MN = 4$. □

Problem 5.5: The lengths in the diagram are as marked, and $\overline{WX} \parallel \overline{YZ}$. Find PY and WX.



Solution for Problem 5.5: Where does this solution go wrong:

Bogus Solution: Since $\overline{WX} \parallel \overline{ZY}$, we have $\angle W = \angle Z$ and $\angle X = \angle Y$. Therefore, $\triangle WPX \sim \triangle YPZ$, and we have

$$\frac{PX}{PZ} = \frac{WX}{YZ} = \frac{WP}{PY}$$

Substitution gives

$$\frac{3}{10} = \frac{WX}{12} = \frac{5}{PY}.$$

We can now easily find $YP = 50/3$ and $WX = 18/5$.

This solution doesn't get the vertex order in the similar triangles right, so it sets up the ratios wrong! $\overline{PX}$ and $\overline{PZ}$ are not corresponding sides. $\overline{PX}$ in $\triangle WPX$ corresponds to $\overline{PY}$ in $\triangle ZPY$ because $\angle W = \angle Z$.

Here's what the solution should look like. Pay close attention to the vertex order in the similarity relationship.

Since $\overline{WX} \parallel \overline{ZY}$, we have $\angle W = \angle Z$ and $\angle X = \angle Y$. Therefore, $\triangle WPX \sim \triangle ZPY$. Hence, we have

$$\frac{PX}{PY} = \frac{WX}{YZ} = \frac{WP}{PZ}.$$

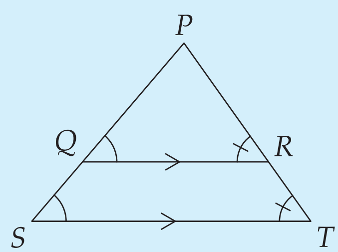
Substitution gives

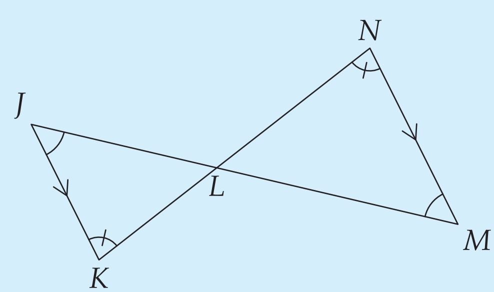
$$\frac{3}{PY} = \frac{WX}{12} = \frac{5}{10}.$$

We can now easily find $PY = 6$ and $WX = 6$. □

Perhaps you see a common thread in the last two problems. While you won't always find parallel lines in similar triangle problems, you'll almost always find similar triangles when you have parallel lines.

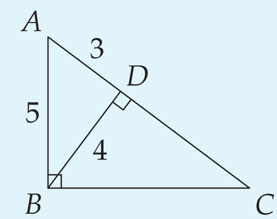
Important: Parallel lines mean equal angles. Equal angles mean similar triangles. The figures below show two very common set-ups in which parallel lines lead to similar triangles. Specifically, $\triangle PQR \sim \triangle PST$ and $\triangle JKL \sim \triangle MNL$.





WARNING!! Read the Bogus Solutions to Problems 5.4 and 5.5 again. These are very common errors; understand them so you can avoid them.

Problem 5.6: Find BC and DC given $AD = 3$, $BD = 4$, and $AB = 5$.



Solution for Problem 5.6: Since $\angle BAD = \angle CAB$ and $\angle BDA = \angle CBA$, we have $\triangle BAD \sim \triangle CAB$ by AA Similarity. Therefore, we have $BC/BD = AB/AD = 5/3$, so $BC = (5/3)(BD) = 20/3$.

We can use this same similarity to find AC, and then subtract AD to get CD. We could also note that $\angle BCD = \angle BCA$ and $\angle BDC = \angle CBA$, so $\triangle BCD \sim \triangle ACB$ by AA Similarity. Therefore, $CD/BD = BC/AB = (20/3)/5 = 4/3$, so $CD = (4/3)(BD) = 16/3$. $\square$

Similar triangles – they're not just for parallel lines.

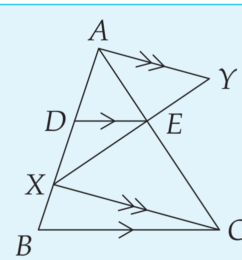
Important: Similar triangles frequently pop up in problems with right angles. The diagram in Problem 5.6 shows a common way this occurs. Make sure you see that

$$\triangle ABD \sim \triangle BCD \sim \triangle ACB.$$

As you'll see throughout the rest of the book, similar triangles occur in all sorts of problems, not just those with parallel lines and perpendicular lines. They're also an important step in many proofs.

Problem 5.7: Given that $\overline{DE} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{XC}$, prove that

$$\frac{EY}{EX} = \frac{AD}{DB}.$$



Solution for Problem 5.7: Parallel lines mean similar triangles. The ratios of side lengths in the problem also suggest we look for similar triangles.

Since $\overline{AY} \parallel \overline{XC}$, we have $\triangle AYE \sim \triangle CXE$. Now we look at what this means for our ratios. From $\triangle AYE \sim \triangle CXE$, we have $EY/EX = AE/EC$. All we have left is to show that $AE/EC = AD/DB$.

Since $\overline{DE} \parallel \overline{BC}$, we have $\triangle ADE \sim \triangle ABC$. Therefore, $AD/AB = AE/AC$, which is almost what we want! We break AB and AC into $AD + DB$ and $AE + EC$, hoping we can do a little algebra to finish:

$$\frac{AD}{AD + DB} = \frac{AE}{AE + EC}$$

If only we could get rid of the AD and AE in the denominators – then we would have $AD/DB = AE/EC$. Fortunately, we can do it. We can flip both fractions:

$$\frac{AD + DB}{AD} = \frac{AE + EC}{AE}.$$

Therefore, $\frac{AD}{AD} + \frac{DB}{AD} = \frac{AE}{AE} + \frac{EC}{AE}$, so $1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$, which gives us

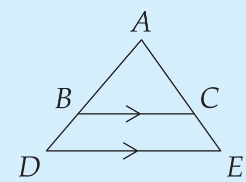
$$\frac{DB}{AD} = \frac{EC}{AE}.$$

Flipping these fractions back over gives us $AD/DB = AE/EC$. Therefore, we have $EY/EX = AE/EC = AD/DB$, as desired. □

Our solution to the previous problem reveals another handy relationship involving similar triangles:

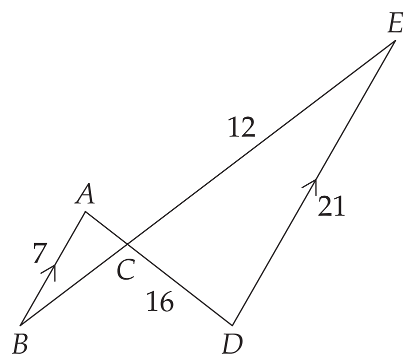
Important: If $\overline{BC} \parallel \overline{DE}$ and $\overleftrightarrow{BD}$ and $\overleftrightarrow{CE}$ meet at A as shown, then

$$\frac{AB}{BD} = \frac{AC}{CE}.$$

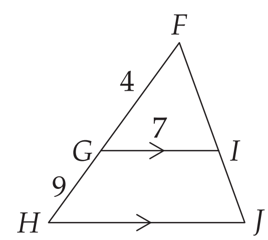


Exercises

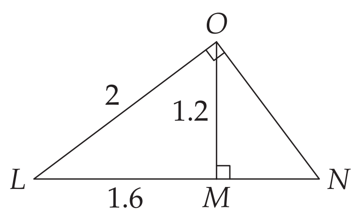
5.2.1



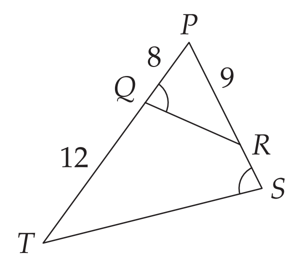
(a) Find AC and BC.



(b) Find HJ.



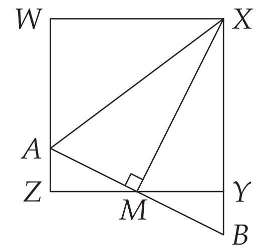
(c) Find ON and MN.



(d) Find RS.

5.2.2 If two isosceles triangles have vertex angles that have the same measure, are the two triangles similar? Why or why not?

5.2.3 In the diagram, $WXYZ$ is a square. M is the midpoint of $\overline{YZ}$, and $\overline{AB} \perp \overline{MX}$.



(a) Show that $\overline{WZ} \parallel \overline{XY}$. **Hints:** 182

(b) Prove that $AZ = YB$.

(c) Prove that $XB = XA$.

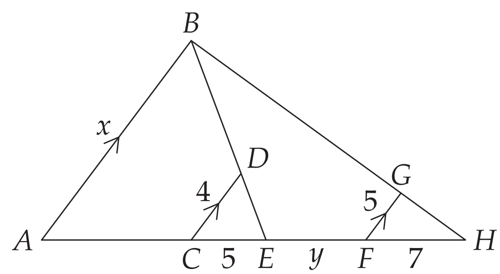
(d) Prove that $\triangle AZM \sim \triangle MYX$, and use this fact to prove $AZ = XY/4$.

5.2.4 In triangle ABC, $AB = AC$, $BC = 1$, and $\angle BAC = 36^\circ$. Let D be the point on side $\overline{AC}$ such that $\angle ABD = \angle CBD$.

(a) Prove that triangles ABC and BCD are similar.

(b)★ Find AB. **Hints:** 150

5.2.5★ Find x in terms of y given the diagram below. **Hints:** 258, 522



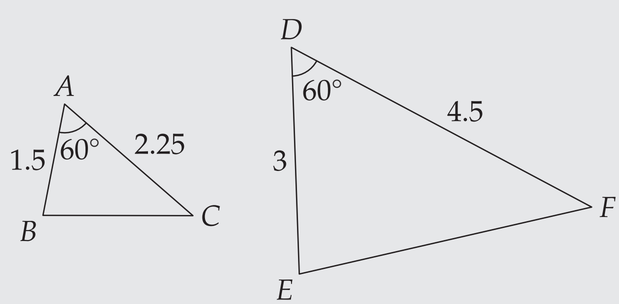
5.3 SAS Similarity

Problems

Problem 5.8:

(a) Measure $\overline{BC}$, $\overline{EF}$, and angles $\angle B$, $\angle C$, $\angle E$, and $\angle F$.

(b) Can you make a guess about how to use Side-Angle-Side for triangle similarity?



Extra! *Descartes commanded the future from his study more than Napoleon from the throne.*
–Oliver Wendell Holmes