Choose always the way that seems the best, however rough it may be; custom will soon render it easy and agreeable. – Pythagoras
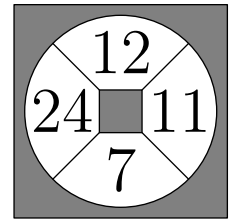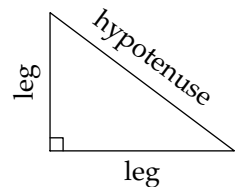
CHAPTER

12

Right Triangles and Quadrilaterals

12.1 The Pythagorean Theorem

In a right triangle, the side of the triangle opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs** of the triangle. We also often use the terms "legs" and "hypotenuse" to refer the lengths of the legs and hypotenuse of a right triangle.
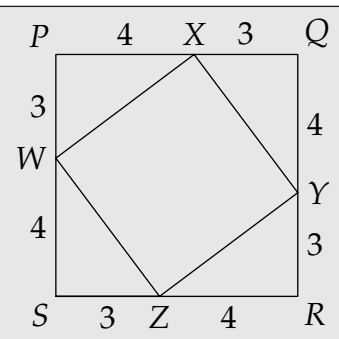

In this section, we explore one of the most famous math theorems, the **Pythagorean Theorem**, which is a powerful relationship among the sides of a right triangle. We'll start by walking through one of the many proofs of the Pythagorean Theorem. ("Pythagorean" is pronounced "puh-thag-uh-ree-uhn.")
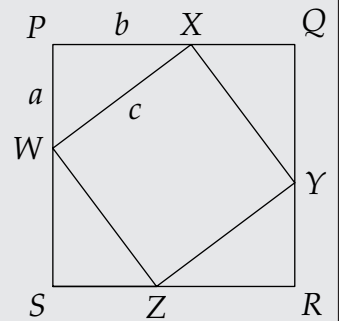
> **Problems**

Problem 12.1: Four identical right triangles with legs of lengths 3 and 4 are attached to the sides of square $WXYZ$ as shown, such that $PW = QX = RY = SZ = 3$ and $PX = QY = RZ = SW = 4$.

(a) Explain why $\angle PWS = 180°$, and why $PQRS$ is a square.

(b) What is the area of $PQRS$?

(c) Find the area of $WXYZ$.

(d) Find WX.

Problem 12.2: In this problem, we follow in the steps of the previous problem to prove the Pythagorean Theorem. We start again with four copies of a right triangle, attached to the sides of square $WXYZ$ as shown at the right. Let the lengths of the legs of each triangle be a and b, as shown, and let the hypotenuse of each right triangle have length c.

(a) Find the area of $WXYZ$ in terms of c.

(b) Find the area of $PQRS$ in terms of a and b.

(c) Find the area of $WXYZ$ in terms of a and b.

(d) Show that $a^2 + b^2 = c^2$.

Problem 12.3: Find the missing side lengths in each of the three triangles shown below.
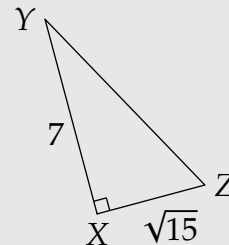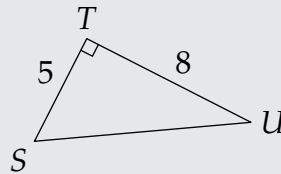
Problem 12.4: Must the hypotenuse of a right triangle be the longest side of the triangle? Why or why not?

Problem 12.5: In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers? **Hints:** 47

Problem 12.6:

(a) Find the hypotenuse of a right triangle whose legs are $3 \cdot 4$ and $4 \cdot 4$.

(b) Find the hypotenuse of a right triangle whose legs are $3 \cdot 5$ and $4 \cdot 5$.

(c) Find the hypotenuse of a right triangle whose legs are $3 \cdot 2011$ and $4 \cdot 2011$.

(d) Find the hypotenuse of a right triangle whose legs are $\frac{3}{100101}$ and $\frac{4}{100101}$.

Problem 12.7: The length of one leg of a right triangle is 210 and the triangle's hypotenuse has length 750. What is the length of the other leg?

CHAPTER 12.   RIGHT TRIANGLES AND QUADRILATERALS

Problem 12.1:   Four identical right triangles with legs of lengths 3 and 4 are attached to the sides of square $WXYZ$ as shown, such that $PW = QX = RY = SZ = 3$ and $PX = QY = RZ = SW = 4$.

(a) Explain why $\angle PWS = 180°$, and why $PQRS$ is a square.

(b) What is the area of $PQRS$?

(c) Find the area of $WXYZ$.

(d) Find WX.
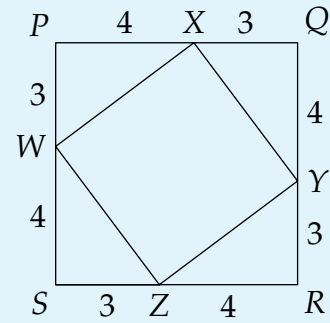
Solution for Problem 12.1:

(a) Back on page 402, we learned that the acute angles of a right triangle add to 90°. Therefore, in right triangle PWX we have

$$\angle PWX + \angle PXW = 90°.$$

Since triangles SWZ and PXW are identical, we have $\angle SWZ = \angle PXW$. Substituting this into the equation above gives
$$\angle PWX + \angle SWZ = 90°.$$
We are told that $WXYZ$ is a square, so $\angle XWZ = 90°$, and we have

$$\begin{aligned}
\angle PWS &= \angle PWX + \angle XWZ + \angle SWZ \\
&= \angle PWX + 90° + \angle SWZ \\
&= 90° + (\angle PWX + \angle SWZ) \\
&= 90° + 90° \\
&= 180°.
\end{aligned}$$

Therefore, $\angle PWS$ is a straight angle. This means that W is on $\overline{PS}$. Similarly, each vertex of $WXYZ$ is on one of the sides of quadrilateral $PQRS$. Each side of $PQRS$ has length $3+4 = 7$, and each angle of $PQRS$ is the right angle of one of the triangles. So, all the sides of $PQRS$ are congruent, and all the angles of $PQRS$ are congruent, which means $PQRS$ is a square.

(b) Since $PQRS$ is a square with side length 7, its area is $7^2 = 49$.

(c) Each right triangle has area $(3)(4)/2 = 6$ square units. Removing the four right triangles from $PQRS$ leaves $WXYZ$, so we have
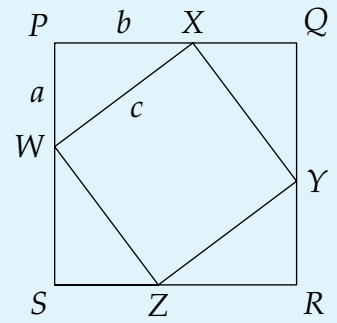
$$[WXYZ] = [PQRS] - 4(6) = 49 - 24 = 25.$$

(d) The area of $WXYZ$ is the square of its side length. Because the area of $WXYZ$ is 25, its side length must be $\sqrt{25}$, which equals 5.

□

Problem 12.2:   In this problem, we follow in the steps of the previous problem to prove the Pythagorean Theorem. We start again with four copies of a right triangle, attached to the sides of square $WXYZ$ as shown at the right. Let the lengths of the legs of each triangle be a and b, as shown, and let the hypotenuse of each right triangle have length c.

(a) Find the area of $WXYZ$ in terms of c.

(b) Find the area of $PQRS$ in terms of a and b.

(c) Find the area of $WXYZ$ in terms of a and b.

(d) Show that $a^2 + b^2 = c^2$.

Solution for Problem 12.2:

(a) Since $WXYZ$ is a square with side length c, its area is c^2.

(b) As in the previous problem, $PQRS$ is a square, and the vertices of $WXYZ$ are on the sides of $PQRS$. Each side of $PQRS$ has length $a + b$, so the area of $PQRS$ is $(a+b)^2$. We can expand $(a + b)^2$ with the distributive property:

$$\begin{aligned}
[PQRS] = (a + b)^2 &= (a + b)(a + b) \\
&= a(a + b) + b(a + b) \\
&= a^2 + ab + ba + b^2 \\
&= a^2 + ab + ab + b^2 \\
&= a^2 + 2ab + b^2.
\end{aligned}$$

(c) The area of each of the right triangles is $ab/2$, so the four right triangles together have area $4(ab/2) = 2ab$. We can find the area of $WXYZ$ in terms of a and b by subtracting the areas of the triangles from the area of $PQRS$:

$$\begin{aligned}
[WXYZ] = [PQRS] - 4(ab/2) \\
= a^2 + 2ab + b^2 - 2ab \\
= a^2 + b^2.
\end{aligned}$$

(d) In part (a), we found that $[WXYZ] = c^2$, and in part (c), we found that $[WXYZ] = a^2 + b^2$. Equating our expressions for $[WXYZ]$, we have
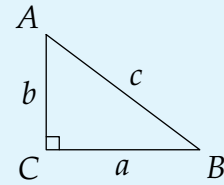
$$a^2 + b^2 = c^2.$$

□

> **Important:** The **Pythagorean Theorem** tells us that in any right triangle, the sum of the squares of the legs equals the square of the hypotenuse. So, in the diagram to the right, we have
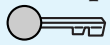>
> $$a^2 + b^2 = c^2.$$

Our work in Problem 12.2 is the same as the work we did in Problem 12.1, except that we replaced the numbers in Problem 12.1 with variables a, b, and c in Problem 12.2.

> **Concept:** Specific examples can sometimes be used as guides to discover proofs.

The Pythagorean Theorem also works "in reverse." By this, we mean that if the side lengths of a triangle satisfy the Pythagorean Theorem, then the triangle must be a right triangle. So, for example, if we have a triangle with side lengths 3, 4, and 5, then we know that the triangle must be a right triangle because $3^2 + 4^2 = 5^2$.

Let's get a little practice using the Pythagorean Theorem.

Problem 12.3: Find the missing side lengths in each of the three triangles shown below.
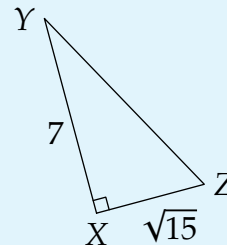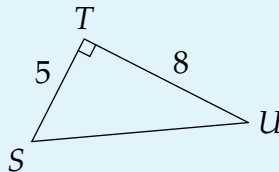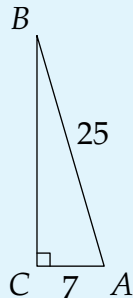
Solution for Problem 12.3: What's wrong with this solution:

> **Bogus Solution:** Applying the Pythagorean Theorem to $\triangle ABC$ gives
>
> $$7^2 + 25^2 = BC^2.$$
>
> Therefore, we find $BC^2 = 49 + 625 = 674$. Taking the square root gives us $BC = \sqrt{674}$.

This solution is incorrect because it applies the Pythagorean Theorem incorrectly. Side $\overline{BC}$ is a leg, not the hypotenuse. Applying the Pythagorean Theorem to $\triangle ABC$ correctly gives

$$AC^2 + BC^2 = AB^2.$$

Substituting $AC = 7$ and $AB = 25$ gives us $7^2 + BC^2 = 25^2$, so $49 + BC^2 = 625$. Subtracting 49 from both sides gives $BC^2 = 576$. Taking the square root of 576 gives $BC = 24$. (Note that $(-24)^2 = 576$ too, but we can't have a negative length, so BC cannot be -24.)
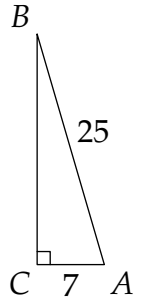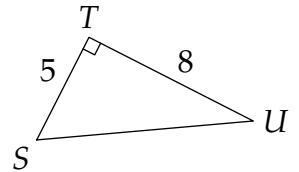
> **WARNING!!** Be careful when applying the Pythagorean Theorem. Make sure you correctly identify which sides are the legs and which is the hypotenuse.

Applying the Pythagorean Theorem to $\triangle STU$ gives
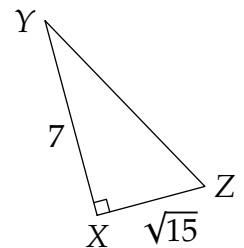
$$ST^2 + TU^2 = SU^2,$$

so we have $5^2 + 8^2 = SU^2$ from the side lengths given in the problem. Therefore, we have $SU^2 = 25 + 64 = 89$. Taking the square root gives us $SU = \sqrt{89}$.

In $\triangle XYZ$, the Pythagorean Theorem gives us

$$XY^2 + XZ^2 = YZ^2,$$

so $7^2 + (\sqrt{15})^2 = YZ^2$. This gives us $49 + 15 = YZ^2$, so $YZ^2 = 64$ and $YZ = 8$. □

> **WARNING!!** A common mistake when using the Pythagorean Theorem to find the hypotenuse length of a right triangle is forgetting that the hypotenuse is squared in the equation, too. One quick way to avoid this error is to consider the three side lengths after finding the hypotenuse.
>
> For example, suppose a right triangle has legs of lengths 3 and 4. The hypotenuse clearly can't be $3^2 + 4^2 = 25$, because the lengths 3, 4, and 25 don't satisfy the Triangle Inequality. Taking the square root of 25, we see that the hypotenuse should be 5, not 25.

Problem 12.4: Must the hypotenuse of a right triangle be the longest side of the triangle? Why or why not?

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Solution for Problem 12.4:   Yes. The square of the hypotenuse equals the sum of the squares of the legs. The sum of any two positive numbers is greater than both of the numbers being added. So, the square of the hypotenuse must be greater than the square of each leg. Therefore, the hypotenuse must be longer than each leg. □

Problem 12.5:   In Problems 12.1 and 12.3, we have seen two right triangles in which all three side lengths are integers. Can you find any more right triangles in which all three side lengths are integers?

Solution for Problem 12.5: There are lots and lots of right triangles in which all three side lengths are integers! To search for some, we can list the first 20 positive perfect squares:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400.$$

Then, we look for pairs of squares that add up to another square. We immediately see $9 + 16 = 25$, which is $3^2 + 4^2 = 5^2$. We already saw this example in Problem 12.1. We also see $25 + 144 = 169$, which is $5^2 + 12^2 = 13^2$. So, a right triangle with legs of lengths 5 and 12 has a hypotenuse with length 13. We also find $64 + 225 = 289$, which is $8^2 + 15^2 = 17^2$. This gives us a right triangle with 8 and 15 as the legs and 17 as the hypotenuse. □

A **Pythagorean triple** is a group of three positive integers that satisfy the equation $a^2 + b^2 = c^2$. So, for example, $\{3, 4, 5\}$ is a Pythagorean triple, as are $\{5, 12, 13\}$ and $\{8, 15, 17\}$. There are lots of interesting patterns in Pythagorean triples. See if you can find more Pythagorean triples, and look for patterns that you can use to find more Pythagorean triples.

We can find one such important pattern by looking back at our list of squares:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400.$$

We find that $36 + 64 = 100$, which is $6^2 + 8^2 = 10^2$. Here, the side lengths are double those of the first triangle we saw with sides of lengths 3, 4, and 5. We might wonder if tripling these three side lengths also gives us another right triangle. Indeed, we see that $9^2 + 12^2 = 15^2$, since $81 + 144 = 225$. Let's investigate further.

Problem 12.6:

(a) Find the hypotenuse of a right triangle whose legs are $3 \cdot 4$ and $4 \cdot 4$.

(b) Find the hypotenuse of a right triangle whose legs are $3 \cdot 5$ and $4 \cdot 5$.

(c) Find the hypotenuse of a right triangle whose legs are $3 \cdot 2011$ and $4 \cdot 2011$.

(d) Find the hypotenuse of a right triangle whose legs are $\frac{3}{100101}$ and $\frac{4}{100101}$.

Solution for Problem 12.6:

(a) The legs have lengths 12 and 16. Letting the hypotenuse be c, the Pythagorean Theorem gives us
$$c^2 = 12^2 + 16^2 = 144 + 256 = 400.$$

Taking the square root gives us $c = 20$. Notice that $20 = 5 \cdot 4$.

(b) The legs have lengths 15 and 20. Letting the hypotenuse be c, the Pythagorean Theorem gives us

$$c^2 = 15^2 + 20^2 = 225 + 400 = 625.$$

Taking the square root gives us $c = 25$. Notice that $25 = 5 \cdot 5$.

(c) The legs have lengths 6033 and 8044. Um, squaring those doesn't look like much fun. Let's see if we can find a more clever way to solve this problem. We know that a right triangle with legs 3 and 4 has hypotenuse 5. In part (a), we saw that if the legs of a right triangle are $3 \cdot 4$ and $4 \cdot 4$, then the hypotenuse is $5 \cdot 4$. In part (b), we saw that if the legs of a right triangle are $3 \cdot 5$ and $4 \cdot 5$, then the hypotenuse is $5 \cdot 5$. It looks like there's a pattern!

> **Concept:** Searching for patterns is a powerful problem-solving strategy.
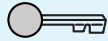
It appears that if the legs of a right triangle are $3x$ and $4x$, then the hypotenuse is $5x$. We can test this guess with the Pythagorean Theorem. Suppose the legs of a right triangle are $3x$ and $4x$. Then, the sum of the squares of the legs is

$$(3x)^2 + (4x)^2 = 3^2x^2 + 4^2x^2 = 9x^2 + 16x^2 = 25x^2.$$

Since

$$(5x)^2 = 5^2x^2 = 25x^2,$$

we have $(3x)^2 + (4x)^2 = (5x)^2$, which means that the length of the hypotenuse is indeed $5x$.

This means that we don't have to square 6033 and 8044! A right triangle with legs of lengths $3 \cdot 2011$ and $4 \cdot 2011$ has a hypotenuse with length $5 \cdot 2011 = 10055$.

(d) There's nothing in our explanation in part (c) that requires x to be a whole number; it can be a fraction, too! So, in a right triangle with legs of length $3 \cdot \frac{1}{100101}$ and $4 \cdot \frac{1}{100101}$, the hypotenuse has length $5 \cdot \frac{1}{100101} = \frac{5}{100101}$.

□

Our work in Problem 12.6 is an example of why knowing common Pythagorean triples is useful. Any time we have a right triangle in which the legs have ratio 3 : 4, then we know that all three sides of the triangle are in the ratio 3 : 4 : 5. As we saw in the final two parts of Problem 12.6, this can allow us to find the hypotenuse quickly without using the Pythagorean Theorem directly.

We can also sometimes use this approach to quickly find the length of a leg when we know the lengths of the other leg and the hypotenuse.

> **Problem 12.7:** The length of one leg of a right triangle is 210 and the triangle's hypotenuse has length 750. What is the length of the other leg?

Solution for Problem 12.7: We find the ratio of the given leg length to the hypotenuse length, hoping it will match the corresponding ratio in one of the Pythagorean triples we know. We have $210 : 750 = \frac{210}{30} : \frac{750}{30} = 7 : 25$, so the ratio of the given leg to the hypotenuse is $7 : 25$. This reminds us of the $\{7, 24, 25\}$ Pythagorean triple that we saw in Problem 12.3. Since the ratio of the known leg to the hypotenuse is $7 : 25$, we know that all three sides are in the ratio $7 : 24 : 25$. The first leg is $7 \cdot 30$ and the hypotenuse is $25 \cdot 30$, so the other leg of the right triangle is $24 \cdot 30 = 720$. $\square$

> **Important:** If we multiply all three side lengths of a right triangle by the same positive number, then the three new side lengths also satisfy the Pythagorean Theorem. In other words, if side lengths a, b, and c satisfy $a^2 + b^2 = c^2$, then $(na)^2 + (nb)^2 = (nc)^2$, for any positive number n.

Exercises

**12.1.1** Find the missing side lengths below:
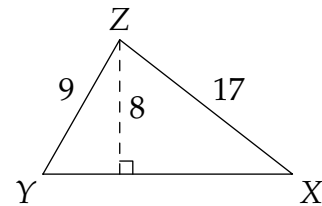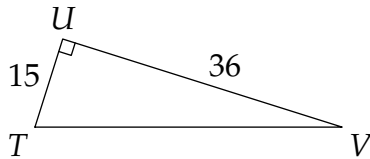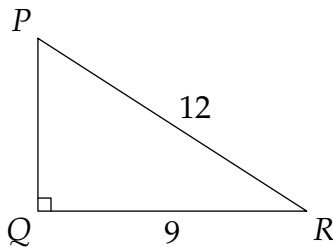
12.1.2 Bill walks $\frac{1}{2}$ mile south, then $\frac{3}{4}$ mile east, and finally $\frac{1}{2}$ mile south. How many miles is he, in a direct line, from his starting point? *(Source: AMC 8)*

12.1.3 Find a formula for the length of a diagonal of a rectangle with length l and width w.

12.1.4 The bases of a 39-foot pole and a 15-foot pole are 45 feet apart, and both poles are perpendicular to the ground. The ground is flat between the two poles. What is the length of the shortest rope that can be used to connect the tops of the two poles?

12.1.5 A square, a rectangle, a right triangle, and a semicircle are combined to form the figure at the right. Find the area of the whole figure in square units.
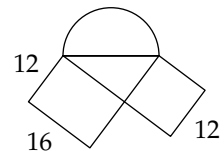
12.1.6★ Find the hypotenuse of a right triangle whose legs have lengths 4900049 and 6300063.