5.2 Solving Linear Equations I

An **equation** states that two quantities are equal. The most basic type of equation comes from arithmetic. For example,

$$2 + 6 = 3 + 5.$$

You've already seen many examples of this sort of equation.

So far in this book, nearly every equation with variables has been used to say that two expressions are equivalent, such as

$$a + b = b + a.$$

In this section, we introduce equations with a variable such that the equation is true for only some values of the variable. Unfortunately, we use the same symbol, "=", to mean that two expressions are equivalent and to write equations that are only true for some values of a variable.

For example, the equation $x + 3 = 9$ does not tell us that $x + 3$ is 9 for all values of x. If $x = 3$, then $x + 3$ is 6, not 9, so the equation $x + 3 = 9$ is not true when $x = 3$. However, if $x = 6$, then $x + 3$ is 9, so the equation $x + 3 = 9$ is true when $x = 6$. The **solutions** to an equation are the values of the variables that make the equation true. So, $x = 6$ is a solution to $x + 3 = 9$.

We say that we **solve** an equation when we find all values of the variable that make the equation true. The two most important tactics we use to solve equations are:

1. *We can replace any expression with an equivalent expression*. For example, in the equation

   $$5x - 4x + 3 = 14,$$

   we can simplify the left-hand side to $x + 3$, so the equation becomes

   $$x + 3 = 14.$$

2. *We can perform the same mathematical operation to both sides of the equation*. For example, starting with the equation $x + 3 = 14$, we can subtract 3 from both sides of the equation to get

   $$x + 3 - 3 = 14 - 3.$$

   Simplifying both sides of the equation then gives $x = 11$, and we have found the solution to the equation. Looking back to the original equation, $5x - 4x + 3 = 14$, we see that when we have $x = 11$, we get $5 \cdot 11 - 4 \cdot 11 + 3 = 14$, which is indeed a true equation.

Important: If you add, subtract, multiply, or divide the expression on one side of the equation by something, then you have to do the same to the expression on the other side of the equation.

We often solve equations with one variable by performing operations on both sides of the equation and simplifying expressions until the variable is alone on one side of the equation. When we do this, we say that we **isolate** the variable.

In this section, we focus on solving **linear equations**. An equation is a linear equation if every term in the equation is a constant term or is a constant times the first power of the variable. So,

$$2x + 4x - 5 = 3 - 6x \qquad \text{and} \qquad 2y + 7 = 3 - 2y$$

are linear equations. The equations

$$x^2 = 36 \qquad \text{and} \qquad \frac{2}{y^3 - 5} = 19$$

are not linear equations.

Problems

Problem 5.8: Consider the equation $x - 12 = 289$. We will solve this equation in several different ways.

(a) Use your understanding of numbers to find a value of x that makes the equation true.

(b) Use the number line to find a value of x that makes the equation true.

(c) What number can be added to both sides of the equation to give an equation in which x is alone on the left side?

(d) Use part (c) to solve the equation.

Problem 5.9: Solve the following equations:

(a) $x - 4\frac{2}{3} = 2\frac{4}{5}$

(b) $4 - 5\frac{1}{5} = 2x + 3 - x + 3\frac{1}{5}$

Problem 5.10: Consider the equation $31x = 713$.

(a) By what number can we divide both sides of the equation to give an equation in which x is alone on the left side?

(b) Solve the equation.

Problem 5.11: Solve the following equations:

(a) $5t = -13$

(b) $24 = -75y$

(c) $\dfrac{u}{7} = \dfrac{3}{14}$

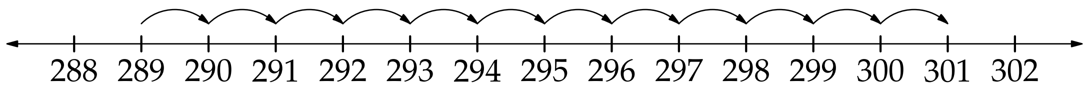
(d) $-\dfrac{2r}{9} = \dfrac{8}{15}$

Problem 5.8: Solve the equation $x - 12 = 289$.

Solution for Problem 5.8: We present three different solutions.

Inspection. The equation means that 12 less than x equals 289. Since 289 is 12 less than x, we know that x must be 12 more than 289. Therefore, x equals $289 + 12$, which is 301.

Number Line. If we consider the number line, the equation $x - 12 = 289$ tells us that 289 is 12 steps to the left of x. This means that x is 12 steps to the right of 289, so x is $289 + 12 = 301$.



Algebra. To solve the equation, we manipulate it until it reads $x =$ (some number). Therefore, we must get x alone on one side of the equation. To do so, we eliminate the -12 on the left side by adding 12 to both sides of the equation:

$$\begin{array}{rcl} x - 12 &=& 289 \\ +\ 12 &=& +12 \\ \hline x \qquad &=& 301 \end{array}$$

We have therefore isolated x on the left side of the equation. We can now see that the solution to the equation $x - 12 = 289$ is $x = 301$.

Whichever method we use to solve the equation, we can check our answer by substituting our solution, $x = 301$, back in to the original equation, $x - 12 = 289$, to get $301 - 12 = 289$. This equation is true, so our solution works. □

Perhaps you noticed that each of our three solution approaches comes down to the same key step, adding 12 to 289 to get our answer. The first uses words, the second uses pictures, the third uses algebra. While logic and pictures are sometimes helpful in solving equations, algebraic manipulations are by far the most generally useful tools to solve equations. Try using algebra to solve the following equations.

Problem 5.9: Solve the following equations:

(a) $x - 4\frac{2}{3} = 2\frac{4}{5}$

(b) $4 - 5\frac{1}{5} = 2x + 3 - x + 3\frac{1}{5}$

Solution for Problem 5.9:

(a) We isolate x by adding $4\frac{2}{3}$ to both sides:

$$\begin{array}{rcl} x - 4\frac{2}{3} &=& 2\frac{4}{5} \\ +\ 4\frac{2}{3} &=& +4\frac{2}{3} \\ \hline x \qquad &=& 2\frac{4}{5} + 4\frac{2}{3} \end{array}$$

We finish by adding the mixed numbers on the right side:

$$x = 2\frac{4}{5} + 4\frac{2}{3} = 2 + 4 + \frac{4}{5} + \frac{2}{3} = 6 + \frac{12}{15} + \frac{10}{15} = 6 + \frac{22}{15} = 6 + 1\frac{7}{15} = 7\frac{7}{15}.$$

This example shows how algebra can help keep our work organized and simple. If we take a logic or picture approach, the fractions might lead to confusion. The algebraic approach makes it very clear how to find the answer.

(b) We start by simplifying both sides of the equation. The left side is simply $4 - 5\frac{1}{5} = -1\frac{1}{5}$. On the right side, we combine the two variable terms and combine the two constants:

$$2x + 3 - x + 3\frac{1}{5} = (2x - x) + \left(3 + 3\frac{1}{5}\right) = x + 6\frac{1}{5}.$$

Now our equation is

$$-1\frac{1}{5} = x + 6\frac{1}{5}$$

To solve this equation, we isolate x by subtracting $6\frac{1}{5}$ from both sides:

$$\begin{array}{rcl} -1\frac{1}{5} &=& x + 6\frac{1}{5} \\ -6\frac{1}{5} &=& \quad - 6\frac{1}{5} \\ \hline -7\frac{2}{5} &=& x \end{array}$$

We typically write the variable first when communicating the solution. The solution to the original equation is $x = -7\frac{2}{5}$.

□

Concept: Isolate, isolate, isolate. The key to solving most equations is to get the variable alone on one side of the equation.

Addition and subtraction are not the only tools we can use to solve linear equations.

Problem 5.10: Solve the equation $31x = 713$.

Solution for Problem 5.10: We divide both sides of the equation by 31. This leaves x alone on the left:

$$\frac{31x}{31} = \frac{713}{31}.$$

Since $31x/31 = x$ and $713/31 = 23$, we have $x = 23$. □

In this solution we used division to change the coefficient of x from 31 to 1. We could also have viewed this as multiplying both sides of the equation by the reciprocal of the coefficient of $31x$ to give $\frac{1}{31} \cdot 31x = \frac{1}{31} \cdot 713$. The $\frac{1}{31}$ and 31 cancel on the left, and we have $x = \frac{713}{31} = 23$.

Problem 5.11: Solve the following equations:

(a) $5t = -13$

(b) $24 = -75y$

(c) $\dfrac{u}{7} = \dfrac{3}{14}$

(d) $-\dfrac{2r}{9} = \dfrac{8}{15}$

Solution for Problem 5.11:

(a) We isolate t by dividing both sides of the equation by 5:

$$\frac{5t}{5} = \frac{-13}{5}.$$

Since $\frac{5t}{5}$ simplifies to t, we have $t = -\frac{13}{5}$ as our solution.

(b) We divide both sides by -75:

$$\frac{24}{-75} = \frac{-75y}{-75},$$

so $\frac{24}{-75} = y$. We usually write the variable first, so we can write this equation as

$$y = \frac{24}{-75}.$$

We finish by simplifying the right-hand side:

$$y = \frac{24}{-75} = -\frac{24}{75} = -\frac{8}{25}.$$

Therefore, the solution is $y = -\frac{8}{25}$.

We can check our answer by substituting $y = -\frac{8}{25}$ in the original equation. We see that $-75 \cdot \left(-\frac{8}{25}\right)$ does equal 24, so our answer is correct.

Important: When solving an equation, we can check our answer by substituting our answer back into the original equation. If the original equation is not satisfied by our answer, then we probably made a mistake and should solve the equation again.

(c) To get rid of the 7 in the denominator on the left side, we multiply both sides by 7:

$$7\left(\frac{u}{7}\right) = 7\left(\frac{3}{14}\right).$$

We have $7\left(\frac{u}{7}\right) = \frac{7u}{7} = u$ and $7\left(\frac{3}{14}\right) = \frac{3}{2}$, so the equation above simplifies to $u = \frac{3}{2}$.

(d) At first, it might look like we can't isolate r with one step. But if we write $-\frac{2r}{9}$ as $\left(-\frac{2}{9}\right)r$, we have

$$\left(-\frac{2}{9}\right)r = \frac{8}{15}.$$

Now, we can isolate r by multiplying both sides of the equation by the reciprocal of the coefficient of r. The reciprocal of $-\frac{2}{9}$ is $-\frac{9}{2}$, and multiplying both sides of the equation by $-\frac{9}{2}$ gives

$$\left(-\frac{9}{2}\right)\left(-\frac{2}{9}\right)r = \left(-\frac{9}{2}\right)\frac{8}{15}.$$

The product of a number and its reciprocal is 1, so the left side simplifies to r, as planned. We therefore have

$$r = \left(-\frac{9}{2}\right)\frac{8}{15} = -\frac{9}{2}\cdot\frac{8}{15} = -\frac{12}{5}.$$

Checking our work, we find that when $r = -\frac{12}{5}$, we have

$$-\frac{2r}{9} = -\frac{2(-12/5)}{9} = -\frac{-24/5}{9} = -\left(-\frac{24}{5\cdot 9}\right) = \frac{24}{45} = \frac{8}{15}.$$

So, the equation is indeed satisfied when $r = -\frac{12}{5}$.

□

Exercises

5.2.1 Solve each of the following equations:

(a) $t + 235 = 137$

(b) $a + \dfrac{7}{9} = \dfrac{-2}{9}$

(c) $-6\dfrac{1}{10} = -14 + c$

(d) $-2y + 2\dfrac{3}{5} + 3y = 1\dfrac{7}{10}$

5.2.2 Solve each of the following equations:

(a) $-7y = 343$

(b) $16x = 3\dfrac{1}{3}$

(c) $\dfrac{x}{5} = \dfrac{6}{7}$

(d) $-\dfrac{5y}{2} = -\dfrac{14}{15}$

5.2.3 Solve the equation $5\dfrac{1}{4} - y = 19\dfrac{3}{4}$.

5.2.4 Solve the equation $\dfrac{x-3}{7} = 2$.

5.2.5 Solve the equation $3(r-7) = 24$.

5.2.6★ Find the value of c such that $x = 2$ is a solution to the equation $\dfrac{x}{c} = 3$.

5.3 Solving Linear Equations II

Problems

Problem 5.12: In this problem, we solve the equation $8t + 9 = 65$.

(a) Isolate the $8t$ by subtracting an appropriate constant from both sides.

(b) Solve the resulting equation for t.

Problem 5.13: In this problem, we solve the equation $7j - 4 + 3j = 6 + 2j - 4j - 8$.

(a) Simplify both sides of the equation by combining like terms.

(b) Add an expression to both sides of your equation from part (a) to give an equation in which no variables are on the right-hand side.

(c) Solve the equation resulting from part (b).

(d) Check your answer! Substitute your value of j into the original equation. If it doesn't work, then do the problem again.

Problem 5.14: Solve the following equations:

(a) $8k - 13\frac{2}{5} = -12\frac{1}{25}$

(b) $4(t - 7) = 3(2t + 3)$

(c) $\dfrac{2r - 7}{9} = 3$

(d) $\dfrac{3x + 4}{5} = \dfrac{2x - 8}{7}$

Problem 5.15: Solve the following equations:

(a) $\dfrac{9}{5} - \dfrac{2x}{3} = \dfrac{6x}{5} + \dfrac{7}{3}$

(b) $\dfrac{4 - 7t}{6} = \dfrac{t}{8} + 2$

Problem 5.16:

(a) Find all values of w that satisfy $5w + 3 - 2w = w - 8 + 2w - 3$.

(b) Find all values of z that satisfy $2z - 8 - 5z = 2 - 3z - 10$.

Problem 5.17: For what value of c do the equations $2y - 5 = 17$ and $cy - 8 = 36$ have the same solution for y?

In the last section, we used addition and subtraction to solve some equations, and used multiplication and division to solve others. To solve most linear equations, however, we'll have to use a combination of these tactics.

Problem 5.12: Solve the equation $8t + 9 = 65$.

Solution for Problem 5.12: This equation doesn't look exactly like any of the equations we already know how to solve. It may not be obvious immediately how to isolate t. However, we can isolate $8t$ by subtracting 9 from both sides:

$$\begin{array}{rcr} 8t + 9 & = & 65 \\ -\ 9 & = & -9 \\ \hline 8t \phantom{{}+9} & = & 56 \end{array}$$

Now we have an equation we know how to solve! We divide both sides by 8 to find $t = 7$.

We can check our work by substituting this value for t back into our original equation. We find that $8(7) + 9 = 65$, so our answer works.

We didn't have to add first when we solved this equation. We could have divided first:

$$\frac{8t + 9}{8} = \frac{65}{8}.$$

We can then distribute on the left side. Since

$$\frac{8t + 9}{8} = \frac{8t}{8} + \frac{9}{8} = t + \frac{9}{8},$$

we have

$$t + \frac{9}{8} = \frac{65}{8}.$$

We then subtract $\frac{9}{8}$ from both sides of this equation to get $t = \frac{65}{8} - \frac{9}{8} = \frac{56}{8} = 7$, as before. □

The equation in Problem 5.12 is not exactly like any of the equations we solved in the previous section. However, we were still able to solve it with the same tools.

Concept: When solving an equation that isn't exactly like an equation you have solved before, try to manipulate it into a form you already know how to deal with.

See if you can apply this strategy to the following problem.

Problem 5.13: Solve the equation $7j - 4 + 3j = 6 + 2j - 4j - 8$.

Solution for Problem 5.13: Our first step is to simplify both sides of the equation. By grouping like terms, the left-hand side of the original equation becomes

$$7j - 4 + 3j = (7j + 3j) - 4 = 10j - 4.$$

The right-hand side of the original equation becomes

$$6 + 2j - 4j - 8 = (2j - 4j) + (6 - 8) = -2j - 2.$$

Combining these results simplifies the original equation to

$$10j - 4 = -2j - 2.$$

We haven't solved any equations in which the variable appears on both sides. We know how to handle an equation if the variable only appears on one side. So, we add $2j$ to both sides to eliminate the variable from the right-hand side:

$$\begin{array}{rcr} 10j - 4 & = & -2j - 2 \\ +\,2j \quad\;\; & = & +2j \quad\;\; \\ \hline 12j - 4 & = & -\,2 \end{array}$$

Now we have an equation we know how to solve! We add 4 to both sides to get $12j = 2$. We then divide by 12 to find $j = \frac{2}{12} = \frac{1}{6}$. □

We now have another strategy for solving linear equations.

> **Concept:** If the variable appears on both sides of the equation, we can use addition and subtraction to get all terms with the variable on the same side of the equation.

Similarly, we use addition and subtraction to get all the constant terms on the other side of the equation.

Here's a little more practice.

> **Problem 5.14:** Solve the following equations:
>
> (a) $8k - 13\frac{2}{5} = -12\frac{1}{25}$
>
> (b) $4(t - 7) = 3(2t + 3)$
>
> (c) $\dfrac{2r - 7}{9} = 3$
>
> (d) $\dfrac{3x + 4}{5} = \dfrac{2x - 8}{7}$

Solution for Problem 5.14:

(a) Adding $13\frac{2}{5}$ to both sides leaves the variable term on the left while putting all the constant terms on the right:

$$8k = -12\frac{1}{25} + 13\frac{2}{5}.$$

Simplifying the right-hand side gives $-12\frac{1}{25}+13\frac{2}{5} = (-12+13)+\left(-\frac{1}{25}+\frac{2}{5}\right) = 1\frac{9}{25}$, so we now have

$$8k = 1\frac{9}{25}.$$

Multiplying both sides by $\frac{1}{8}$ (which is the same as dividing both sides by 8) gives

$$k = \frac{1}{8}\cdot 1\frac{9}{25} = \frac{1}{8}\cdot\frac{34}{25} = \frac{34}{200} = \frac{17}{100}.$$

(b) First, we use the distributive property to expand both sides:

$$4\cdot t - 4\cdot 7 = 3\cdot 2t + 3\cdot 3.$$

Simplifying both sides gives

$$4t - 28 = 6t + 9.$$

Next, we get all the terms with t on one side of the equation and all the constants on the other side. Subtracting $4t$ from both sides gives $-28 = 2t+9$. Subtracting 9 from both sides gives $-37 = 2t$. Finally, dividing both sides by 2 gives $t = -\frac{37}{2}$.

(c) First, make sure you see why adding 7 to $\frac{2r-7}{9}$ doesn't "cancel the -7." This is because $\frac{2r-7}{9}+7$ equals $\frac{2r}{9}-\frac{7}{9}+7$, which is $\frac{2r}{9}+\frac{56}{9}$. There's still a constant term; the $\frac{2r}{9}$ term is not yet isolated.

Since $\frac{2r-7}{9}$ equals $\frac{2r}{9}-\frac{7}{9}$, we add $\frac{7}{9}$ to both sides of

$$\frac{2r}{9}-\frac{7}{9} = 3$$

to eliminate the constant on the left side and isolate $\frac{2r}{9}$. Doing so gives us

$$\frac{2r}{9} = 3+\frac{7}{9} = \frac{34}{9}.$$

Multiplying both sides of $\frac{2r}{9} = \frac{34}{9}$ by $\frac{9}{2}$ gives $r = \frac{34}{9}\cdot\frac{9}{2} = 17$.

We could have avoided fractions entirely by multiplying both sides of $\frac{2r-7}{9} = 3$ by 9 on the first step to get $9\cdot\frac{2r-7}{9} = 27$. Since

$$9\cdot\frac{2r-7}{9} = \frac{9(2r-7)}{9} = \frac{9}{9}(2r-7) = 2r-7,$$

the 9's cancel on the left side of $9\cdot\frac{2r-7}{9} = 27$ to leave $2r-7 = 27$. Adding 7 to both sides gives $2r = 34$, so $r = 17$, as before.

Checking our answer, we find that if $r = 17$, then $\frac{2r-7}{9} = \frac{2\cdot 17-7}{9} = \frac{27}{9} = 3$, as required.

(d) We start by getting rid of the fractions. We eliminate the denominator on the right by multiplying both sides by 7:
$$7\cdot\frac{3x+4}{5} = 7\cdot\frac{2x-8}{7}.$$
The 7's on the right-hand side cancel, because
$$7\cdot\frac{2x-8}{7} = \frac{7\cdot(2x-8)}{7} = \frac{7}{7}\cdot\frac{2x-8}{1} = 2x-8.$$
So, we can write $7\cdot\frac{3x+4}{5} = 7\cdot\frac{2x-8}{7}$ as
$$\frac{7(3x+4)}{5} = 2x-8.$$
Next, we multiply both sides by 5 to cancel the 5 in the denominator on the left-hand side:
$$5\cdot\frac{7(3x+4)}{5} = 5(2x-8).$$
The 5's on the left cancel, and we are left with
$$7(3x+4) = 5(2x-8).$$
Expanding both sides gives
$$7(3x)+7(4) = 5(2x)-5(8).$$
Simplifying both sides gives $21x+28 = 10x-40$, and now we're in familiar territory. Subtracting $10x$ from both sides gives $11x+28 = -40$. Subtracting 28 from both sides gives $11x = -68$. Dividing both sides by 11 gives $x = -\frac{68}{11}$.

□

Notice that multiplying both sides of
$$\frac{3x+4}{5} = \frac{2x-8}{7}$$
by the denominators of both fractions gave us
$$7(3x+4) = 5(2x-8).$$
Rather than performing these multiplications as two separate steps, we will often perform both at once. Multiplying both sides of the original equation by 5 and by 7 gives
$$5\cdot 7\cdot\frac{3x+4}{5} = 5\cdot 7\cdot\frac{2x-8}{7}.$$

The 5 on the left cancels with the 5 in the denominator on the left, and the 7 on the right cancels with the 7 in the denominator on the right, leaving

$$7(3x + 4) = 5(2x - 8).$$

We call this process **cross-multiplying**.

Our last example above showed another way to simplify working with equations:

Concept: If you don't like dealing with fractions, you can eliminate fractions from a linear equation by multiplying both sides of the equation by a constant that cancels the denominators of the fractions.

Let's practice this strategy.

Problem 5.15: Solve the following equations:

(a) $\frac{9}{5} - \frac{2x}{3} = \frac{6x}{5} + \frac{7}{3}$

(b) $\frac{4 - 7t}{6} = \frac{t}{8} + 2$

Solution for Problem 5.15:

(a) Let's get rid of the fractions right away. We multiply both sides of the equation by 3 to cancel the denominators that are 3, and multiply by 5 to cancel the denominators that are 5. Therefore, we can take care of both at once by multiplying by $3 \cdot 5 = 15$. Using the distributive property to expand, the left-hand side becomes

$$\begin{aligned} 15\left(\frac{9}{5} - \frac{2x}{3}\right) &= 15 \cdot \frac{9}{5} - 15 \cdot \frac{2x}{3} \\ &= \frac{15}{5} \cdot 9 - \frac{15}{3} \cdot 2x \\ &= 27 - 5 \cdot 2x \\ &= 27 - 10x. \end{aligned}$$

Multiplying the right-hand side of the original equation by 15 gives

$$\begin{aligned} 15\left(\frac{6x}{5} + \frac{7}{3}\right) &= 15 \cdot \frac{6x}{5} + 15 \cdot \frac{7}{3} \\ &= \frac{15}{5} \cdot 6x + \frac{15}{3} \cdot 7 \\ &= 3 \cdot 6x + 5 \cdot 7 \\ &= 18x + 35. \end{aligned}$$

Combining this with our simplified left-hand side gives

$$27 - 10x = 18x + 35.$$

We add $10x$ to both sides to get $27 = 28x+35$. We subtract 35 from both sides to get $-8 = 28x$ and divide by 28 to find $x = -\frac{8}{28} = -\frac{2}{7}$.

(b) We might start by multiplying both sides by $6 \cdot 8$ to cancel both denominators. However, since $\operatorname{lcm}[6,8] = 24$, we can cancel both denominators by multiplying both sides by 24 instead of 48:

$$24\left(\frac{4-7t}{6}\right) = 24\left(\frac{t}{8}+2\right).$$

Multiplying on the left-hand side and distributing on the right gives

$$\frac{24(4-7t)}{6} = 24 \cdot \frac{t}{8} + 24 \cdot 2,$$

so

$$\frac{24}{6}(4-7t) = \frac{24}{8}t + 48.$$

Dividing gives $4(4-7t) = 3t + 48$. No more fractions! Expanding the left-hand side gives us $16 - 28t = 3t + 48$. Adding $28t$ to both sides and subtracting 48 from both sides gives $-32 = 31t$. Dividing by 31 gives us $t = -\frac{32}{31}$.

□

So far, all the equations we have solved have had exactly one solution. This isn't always the case!

Problem 5.16:

(a) Find all values of w that satisfy $5w + 3 - 2w = w - 8 + 2w - 3$.

(b) Find all values of z that satisfy $2z - 8 - 5z = 2 - 3z - 10$.

Solution for Problem 5.16:

(a) We first simplify both sides. This gives us

$$3w + 3 = 3w - 11.$$

When we next try to get all the w terms on one side by subtracting $3w$ from both sides, we have

$$3 = -11.$$

Uh-oh! What happened to the w's? They all canceled. Worse yet, we are left with an equation that can clearly never be true, since 3 cannot ever equal -11!

Since the equation $3 = -11$ can never be true, we know that the original equation can never be true either. That is, the original equation is not true for any value of w. We can see why when we look back to the equation $3w + 3 = 3w - 11$. The left-hand side is 14 greater than the right-hand side, no matter what value of w we use.

We conclude that there are no solutions to the original equation.

(b) Once again, we simplify both sides of the equation, which gives

$$-3z - 8 = -3z - 8.$$

Since both sides of the equation simplify to the same expression, we see that the equation is *always* true! No matter what value of z we choose, the equation will always be true. Therefore, all values of z satisfy the given equation.

□

We see now that some linear equations have no solutions, and others that are satisfied by every value of the variable in the equation.

Important: If a linear equation can be manipulated into an equation that is never true (such as $3 = -11$), then there are no solutions to the equation.

If the two sides of an equation are equivalent, such as in the equation $-3z - 8 = -3z - 8$, then all possible values of the variable are solutions to the original equation. Similarly, if a linear equation can be manipulated into an equation in which both sides are identical, then all possible values of the variable are solutions to the original equation. (The one exception to this is if one of the manipulations is multiplying both sides by 0, which is a pretty silly thing to do to a linear equation!)

Problem 5.17: For what value of c do the equations $2y - 5 = 17$ and $cy - 8 = 36$ have the same solution for y?

Solution for Problem 5.17: We know how to handle the first equation, so let's start there. By solving the first equation for y, we can find the value of y that must satisfy both equations. Adding 5 to both sides of $2y - 5 = 17$ gives $2y = 22$. Dividing by 2 then gives $y = 11$. This value of y must also satisfy $cy - 8 = 36$. So, when we substitute $y = 11$ into $cy - 8 = 36$, we must have a true equation. This substitution gives

$$11c - 8 = 36.$$

Now that we have a linear equation for c, we can find c. Adding 8 to both sides gives $11c = 44$. Dividing by 11 then gives $c = 4$. □

Extra! *Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean.* –G. H. Hardy

Exercises

5.3.1 Solve the following equations:

(a) $2x + 5 = 11$ (b) $\frac{1}{3} = -1\frac{1}{2} - 6a$ (c) $-7t + 19 = 61$

5.3.2 Solve the following equations:

(a) $3y + 9 = 2y + 1$ (b) $5x - 3 - x = 14 - 3x + 11$ (c) $1000a + 218 = 998a + 232$

5.3.3 If $3x - 2 = 11$, then what is the value of $6x + 5$?

5.3.4 Solve the following equations:

(a) $\frac{2}{3}t + \frac{4}{5} = -\frac{1}{2}$ (b) $\frac{1}{2}(z + 3) = \frac{1}{3}(z - 7)$ (c) $\frac{4x}{7} - \frac{1}{2} = -\frac{3}{4} - \frac{2x}{5}$

5.3.5 Solve $\frac{2x + 7}{5} = -\frac{1 - 3x}{8}$.

5.3.6 Solve the following equations:

(a) $2(z + 3) - 5(6 - z) = 8(3z + 3) - 4(1 - 2z)$

(b) $\frac{m + 11}{3} + \frac{m - 2}{6} = \frac{2m - 1}{12}$

(c) $\frac{p - 2}{4} = \frac{2p - 3}{8}$

5.4 Word Problems

Most word problems can be solved using the following general method:

1. Read the problem carefully. Wait, I didn't say that loud enough:

Read the problem carefully!

2. Convert the words to math.
3. Solve the math.
4. Convert your answer back to words.
5. Check your answer (and check to be sure that you answered the question that was asked).