Exercises

16.4.1 If f is a function that has an inverse and $f(3) = 5$, what is $f^{-1}(5)$?

16.4.2 Find the inverse of each of the following functions, if it exists. If the function does not have an inverse, explain why.

(a) $f(x) = 3x + 2$

(b) $f(x) = 13$

(c) $f(x) = \dfrac{4x-5}{x-4}$

(d) $f(x) = 2x^2 + 3$

(e) $f(x) = x^3$

(f) $f(x) = \dfrac{1}{2x}$

16.4.3★ For what values of a is the function $f(x) = \dfrac{x}{x-a}$ its own inverse?

16.4.4★ In one step of our first solution to Problem 16.16, we divide by $2 - x$. This is only valid if $x \neq 2$. Why can we be sure that x cannot be equal to 2? **Hints:** 179

16.5 Problem Solving with Functions

We've seen thus far that solving basic problems involving functions is typically a matter of substitution and solving equations. The same is true as the problems get more challenging.

Problems

Problem 16.18: If $f(x-3) = 9x^2 + 2$, what is $f(5)$?

Problem 16.19: Let f be a function for which $f(x/3) = x^2 + x + 1$. In this problem we find the sum of all values of z for which $f(3z) = 7$. *(Source: AMC 12)*

(a) What must we let x equal in order to use our definition of f to get an expression for $f(3z)$?

(b) Make the substitution suggested by part (a) to produce an equation. Find the sum of the values of z that satisfy this equation.

Problem 16.20: Daesun starts counting at 100, and he counts by fours: 100, 104, 108, ... Andrew starts counting at 800, and he counts backwards by three: 800, 797, 794, ... They both start counting at 1 PM, and each says one number each minute. What time is it when Daesun first says a number that is more than twice the number Andrew says?

(a) Let $D(x)$ be the number Daesun says x minutes after 1 PM. In terms of x, what is $D(x)$?

(b) Let $A(x)$ be the number Andrew says x minutes after 1 PM. In terms of x, what is $A(x)$?

(c) Write an inequality for how $D(x)$ and $A(x)$ are related when Daesun says a number that is more than twice the number Andrew says.

(d) Find the desired time.

Problem 16.21: A function f defined for all positive integers has the property that $f(m) + f(n) = f(mn)$ for any positive integers m and n. If $f(2) = 7$ and $f(3) = 10$, then calculate $f(12)$. *(Source: Mandelbrot)*

Problem 16.22: The function f has the property that, whenever a, b, and n are positive integers such that $a + b = 2^n$, then $f(a) + f(b) = n^2$.

(a) Let $a = b = 1$ to find $f(1)$.

(b) Find $f(2)$, $f(4)$, $f(8)$, and $f(16)$.

(c) Find $f(2^k)$ in terms of k.

(d) Find $f(3)$.

(e) What is $f(2002)$? *(Source: HMMT)*

While many function problems require substitution to solve them, we have to be careful about what we are substituting.

Problem 16.18: If $f(x - 3) = 9x^2 + 2$, what is $f(5)$?

Solution for Problem 16.18: What's wrong with this solution:

Bogus Solution:

$$f(5) = 9(5^2) + 2 = 227.$$

This Bogus Solution assumes that $f(x) = 9x^2 + 2$, but that's not true! The input to the function in the function definition is $x - 3$, not x.

Solution 1: Find the correct x. One way to find $f(5)$ is to find the x that allows us to input 5 into f using the definition of $f(x - 3)$. Solving $x - 3 = 5$ gives $x = 8$. If we let $x = 8$ in our function definition, we find

$$f(8 - 3) = 9(8^2) + 2,$$

from which we get $f(5) = 578$.

Solution 2: Find $f(x)$. We can turn $f(x-3)$ into $f(x)$ by choosing the proper expression for x. Specifically, if we let $z = x - 3$, we have $x = z + 3$. Substituting this into our function definition, we have

$$f(z + 3 - 3) = 9(z + 3)^2 + 2,$$

so $f(z) = 9(z + 3)^2 + 2$. The z is just a dummy variable, so we can freely change it to whatever letter we want, like x:

$$f(x) = 9(x + 3)^2 + 2.$$

So, $f(5) = 9(5 + 3)^2 + 2 = 578$, as before. □

Equations involving functions such as $f(x - 3) = 9x^2 + 2$ are sometimes called **functional equations**. As we have seen, when we substitute for variables in a functional equation, we must be careful to substitute properly for that variable everywhere.

Problem 16.19: Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$. *(Source: AMC 12)*

Solution for Problem 16.19: In order to turn $f(3z) = 7$ into an equation for z, we must find an expression for $f(3z)$. We have an expression for $f(x/3)$, so if we turn $x/3$ into $3z$, we'll have the desired $f(3z)$. If $x/3 = 3z$, then $x = 9z$. Substituting $x = 9z$ into

$$f(x/3) = x^2 + x + 1.$$

gives

$$f(9z/3) = (9z)^2 + 9z + 1,$$

so $f(3z) = 81z^2 + 9z + 1$. Therefore, the equation $f(3z) = 7$ becomes

$$81z^2 + 9z + 1 = 7,$$

so $81z^2 + 9z - 6 = 0$. The sum of the roots of this quadratic is $-(9/81) = -1/9$. □

We can define functions to help solve word problems in the same way we define variables to help us.

Problem 16.20: Daesun starts counting at 100, and he counts by fours: 100, 104, 108, ... Andrew starts counting at 800, and he counts backwards by three: 800, 797, 794, ... They both start counting at 1 PM, and say one number each minute. What time is it when Daesun first says a number that is more than twice the number Andrew says?

Solution for Problem 16.20: In order to compare Daesun's number to Andrew's, we need an expression for each in terms of the time. So, we define a function, $D(x)$, for Daesun, and a function, $A(x)$, for Andrew:

Let $D(x)$ be Daesun's number x minutes after 1 PM.
Let $A(x)$ be Andrew's number x minutes after 1 PM.

Since Daesun starts at 100 and counts up by fours, we have

$$D(x) = 100 + 4x.$$

Since Andrew starts a 800 and counts down by threes, we have

$$A(x) = 800 - 3x.$$

We seek the first time such that

$$D(x) > 2A(x).$$

Our expressions for $D(x)$ and $A(x)$ give us

$$100 + 4x > 2(800 - 3x).$$

Solving this inequality gives us $x > 150$. The smallest such x is 151, so the first time Daesun says a number that is more than twice Andrew's number is 151 minutes after 1 PM, or 3:31 PM. □

Concept: Defining functions is a good way to organize information.

Our final two problems involve functional equations in which we seek a specific value of a function given more complicated information involving the function. Just as with our earlier problems, clever substitution is the key to solving these problems.

Problem 16.21: A function f defined for all positive integers has the property that $f(m)+f(n) = f(mn)$ for any positive integers m and n. If $f(2) = 7$ and $f(3) = 10$, then calculate $f(12)$. *(Source: Mandelbrot)*

Solution for Problem 16.21: We start by experimenting with the information we have. From the equation

$$f(m) + f(n) = f(mn),$$

we see that if we know $f(m)$ and $f(n)$, then we know $f(mn)$. Since we know $f(2)$ and $f(3)$, we know $f(6)$:

$$f(6) = f(2) + f(3) = 17.$$

We want $f(12)$. Since $12 = 2 \cdot 6$ and we know both $f(2)$ and $f(6)$, we can find $f(12)$:

$$f(12) = f(2) + f(6) = 7 + 17 = 24.$$

See if you can find another solution by first finding $f(4)$. □

Concept: Many complicated-looking functional equation problems can be solved with a little experimentation. Don't let the notation scare you; these problems are often not nearly as hard as they look!

Problem 16.22: The function f has the property that, whenever a, b, and n are positive integers such that $a + b = 2^n$, then $f(a) + f(b) = n^2$. What is $f(2002)$? *(Source: HMMT)*

Solution for Problem 16.22: Here we aren't given any values of f, but we have to find $f(2002)$. So, we start by trying to find some values of $f(m)$ for various integers m. We start at the beginning.

Concept: Start experimenting with functional equations by trying simple values like 0 and 1.

We choose simple values of a, b, and n that satisfy

$$a + b = 2^n.$$

The simplest is $a = 1$, $b = 1$, and $n = 1$. Since $1 + 1 = 2^1$, we are told that

$$f(1) + f(1) = 1^2.$$

Therefore, $f(1) = 1^2/2 = 1/2$. We found one value of $f(m)$! But we're still pretty far from finding $f(2002)$. However, this simple example suggests a way to find some more values for $f(m)$. Since $2 + 2 = 2^2$, we have

$$f(2) + f(2) = 2^2 = 4.$$

So, $f(2) = 2^2/2 = 2$. Similarly, $4 + 4 = 2^3$, so

$$f(4) + f(4) = 3^2 = 9,$$

and $f(4) = 3^2/2 = 9/2$. In this same way, we find $f(8) = 4^2/2 = 8$, $f(16) = 5^2/2 = 25/2$, and so on. We can prove that this pattern always works. For each power of 2, we have

$$2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

Therefore, we have

$$f(2^k) + f(2^k) = (k+1)^2,$$

so $f(2^k) = (k+1)^2/2$.

Concept: A great deal of problem solving follows the process:

Experiment → Find Pattern → Prove Pattern is True.

Almost all great discoveries have their beginnings in experimentation.

But how do we find $f(m)$ if m is not a power of 2? Let's try experimenting again by trying to find $f(3)$. We must have $a + 3 = 2^n$ in order to be able to use $f(a) + f(3) = n^2$ to find $f(3)$. Furthermore, we must know $f(a)$, since we can't let $a = 3$. Fortunately, $a = 1$ fits the bill: $1 + 3 = 2^2$, so

$$f(1) + f(3) = 2^2 = 4.$$

We already have $f(1) = 1/2$, so $f(3) = 4 - f(1) = 7/2$.

But how does this help with $f(2002)$? It gives us some guidance: we see that we need a number a such that $a + 2002 = 2^n$. The smallest such number is 46:

$$46 + 2002 = 2^{11}.$$

So, we know that $f(46) + f(2002) = 11^2 = 121$, from which we have

$$f(2002) = 121 - f(46).$$

Unfortunately, we don't know $f(46)$. However, if we find $f(46)$, then we can find $f(2002)$, so we've reduced our problem from finding $f(2002)$ to finding $f(46)$. This appears to be a simpler problem.

Concept: Keep your eye on the ball! Working backwards from what you want to find is a great way to solve problems.

We investigate $f(46)$ just as we investigated $f(2002)$. Since $46+18 = 2^6$, we have $f(46)+f(18) = 6^2 = 36$, so

$$f(46) = 36 - f(18),$$

and we've reduced our problem to finding $f(18)$. This is promising, so we continue.

Since $18 + 14 = 2^5$, we have $f(18) + f(14) = 25$, so $f(18) = 25 - f(14)$.

Since $14 + 2 = 2^4$, we have $f(14) + f(2) = 4^2 = 16$, so $f(14) = 16 - f(2)$. But we already know $f(2) = 2$! We have $f(14) = 16 - 2 = 14$. Now we can work back through our equations above to find $f(2002)$.

We have $f(18) = 25 - f(14) = 11$, so $f(46) = 36 - f(18) = 25$, so $f(2002) = 121 - f(46) = 96$. □

This problem highlighted two of the most important problem solving strategies: experimentation and working backwards. Try them on the following problems whenever you get stuck.

Exercises

16.5.1 Let $g(2x + 5) = 4x^2 - 3x + 2$. Find $g(-3)$.

16.5.2 Alice and Bob go for a run in the local park. Alice runs at 3 m/s. Bob starts from the same point as Alice, but he starts 20 seconds after Alice. Bob runs at a rate of 5 m/s.

(a) Let t be the number of seconds that have elapsed since Bob started running. Find functions describing Alice's and Bob's distance in meters from Bob's starting position in terms of t.

(b) How many seconds after Bob starts running has he run 50% farther than Alice?

16.5.3 If $f(2x) = \dfrac{2}{2+x}$ for all $x > 0$, then what is $2f(x)$? *(Source: AHSME)*

16.5.4 Let $P(n)$ and $S(n)$ denote the product and the sum, respectively, of the digits of the integer n. For example, $P(23) = 6$ and $S(23) = 5$. Suppose N is a two-digit number such that $N = P(N) + S(N)$. What is the units digit of N? *(Source: AMC 12)* **Hints:** 155

16.5.5★ A function $f(x, y)$ of two variables has the property that

$$f(x, y) = x + f(x - 1, x - y).$$

If $f(1, 0) = 5$, then what is the value of $f(5, 2)$? *(Source: Mandelbrot)* **Hints:** 191

16.6 Operations

You're already familiar with several operations. For example, the operation "+" tells us to find the sum of two numbers, and the operation "×" tells us to find the product of them. Operations work just like functions do because operations essentially are functions of two variables. The operations are just written with a different notation, usually because what we're doing with the operation is so common that we want simpler notation than functions offer. So, instead of writing $+(3, 5)$ to mean "3 plus 5," we write $3 + 5$.