Exercises

7.3.1 Five chickens eat 10 bags of scratch in 20 days. How long does it take 18 chickens to eat 100 bags of scratch?

7.3.2 Suppose a is jointly proportional to b and c. If $a = 4$ when $b = 8$ and $c = 9$, then what is a when $b = 2$ and $c = 18$?

7.3.3★ The force of the gravitational attraction between two bodies is directly proportional to the mass of each body and inversely proportional to the square of distance between them. If the distance between two bodies is tripled and the mass of each is doubled, what happens to the force of gravitational attraction between them? **Hints:** 196

7.4 Rate Problems

As we mentioned at the end of the last section, one of the most common applications of joint proportions is to problems involving rates. Often the rates involved are rates of travel, but they can also be rates of performing any measurable task.

Problems

Problem 7.10: Prima has a dentist appointment today at 11:00. Usually she drives 45 miles per hour and it takes her 40 minutes to get to the dentist from her house. If she leaves her house at 10:30 instead of the usual 10:20, how fast should she drive to get to the dentist's office on time?

Problem 7.11: Jack drove 30 miles per hour to work. As soon as he got to work, he remembered that he forgot to feed his dogs. So, he sped back home, driving 45 miles per hour. In this problem we find his average speed during his round trip to work and back home. By *average speed*, we mean the constant speed that Jack would have to drive both to and from work to complete the trip in exactly the same time as he does driving 30 mph there and 45 mph back.

(a) Let the distance from his home to work be d. In terms of d, how long does it take Jack to drive to work?

(b) In terms of d, how long does it take Jack to drive home from work?

(c) What is the total distance for his round trip?

(d) What is Jack's average speed during his round trip?

Problem 7.12: Pippin and Sam are painting a fence. Sam could paint the whole fence alone in 12 hours. Pippin could paint the whole fence alone in 8 hours. Sam starts painting at 1 p.m. and Pippin joins him at 3 p.m. In this problem we determine at what time they finish.

(a) What portion of the fence does Sam paint in an hour? How about Pippin?

(b) At what time do they finish?

Problem 7.13: Flo and Carl each must read a 500-page book. Flo reads one page every minute. Carl reads one page every 50 seconds. Flo starts reading at 1:00, and Carl starts reading at 1:30. When will Carl catch up to Flo?

Problem 7.14: Barry owns a house on a river. The river flows at 3 miles per hour. Barry decides to start rowing downstream, with the current, at noon. He wants to return to his house at 5 p.m. At what time should Barry turn around and row home if he normally rows 5 miles per hour in water that has no current?

Problem 7.15: Two trains, each moving at 20 miles per hour towards each other, are initially 60 miles apart. A bee starts at the front of one train, flies to the other train, then back to the first train, and so on. If the bee always flies at 30 miles per hour, how far does the bee fly before the trains collide? **Hints:** 7

Problem 7.16: A man is running through a train tunnel. When he is $\frac{2}{5}$ of the way through, he hears a train that is approaching the tunnel from behind him at a speed of 60 mph. Whether he runs ahead or runs back, he will reach an end of the tunnel at the same time the train reaches that end. At what rate, in mph, is he running? *(Source: MATHCOUNTS)* **Hints:** 43

Whenever you travel somewhere and try to figure out how long the journey will take given the distance you have to travel and the speed you are driving, you are using proportions.

Problem 7.10: Prima has a dentist appointment today at 11:00. Usually, she drives 45 miles per hour and it takes her 40 minutes to get to the dentist from her house. If she leaves her house at 10:30 instead of the usual 10:20, how fast should she drive to get to the dentist's office on time?

Solution for Problem 7.10: If we know how fast Prima drives and how long she drives, we can find the distance by simply multiplying the two.

In this problem, the distance is constant. Therefore, Prima's rate, r, and the time she drives, t, are inversely proportional, since rt equals some constant distance. From here, we can solve the problem in two ways:

Solution 1: Find the Distance. She usually drives 45 miles per hour in 40 minutes. We have both minutes and hours among our units. We therefore convert the minutes to hours, so that all the time units in the problem will be the same. Forty minutes is equivalent to $40/60 = 2/3$ hour, so when Prima drives 45 miles per hour for 40 minutes, she drives

$$\left(45\,\frac{\text{miles}}{\cancel{\text{hour}}}\right)\left(\frac{2}{3}\,\cancel{\text{hours}}\right) = 30 \text{ miles}.$$

Now, she has only 30 minutes, or 1/2 hour, to cover the 30 miles. Therefore, her rate must be

$$r = \frac{d}{t} = \frac{30 \text{ miles}}{\frac{1}{2} \text{ hour}} = 60\,\frac{\text{miles}}{\text{hour}}.$$

Solution 2: Use Proportionality. Since the distance is constant, when we multiply the time of driving by $30/40 = 3/4$, we must divide the rate of driving by $3/4$ to keep the product (rate) × (time) constant:

$$\text{New rate} = \frac{\text{Old rate}}{\frac{3}{4}} = \frac{45 \text{ miles per hour}}{\frac{3}{4}} = 60\ \frac{\text{miles}}{\text{hour}}.$$

□

WARNING!! Notice that in both solutions we keep careful track of our units, and that we converted minutes to hours to make all the time units in our equations the same.

As we saw in the last problem, distance traveled equals the product of the rate traveled and the time traveled. A great many problems can be solved by applying this relationship.

Problem 7.11: Jack drove 30 miles per hour to work. As soon as he got to work, he remembered that he forgot to feed his dogs. So, he sped back home, driving 45 miles per hour. What was his average speed during his round trip to work and back home?

Solution for Problem 7.11: What's wrong with this common Bogus Solution:

Bogus Solution: He drove 30 mph (miles per hour) there and 45 mph back, so his average speed is $(30+45)/2 = 37.5$ mph.

This looks convincing; let's see if it passes the "Rate times time equals distance" test. Suppose the distance Jack travels is x miles. Then the time he spends driving to work is

$$\text{Time to work} = \frac{\text{Distance to work}}{\text{Rate to work}} = \frac{x \text{ miles}}{30 \text{ mph}} = \frac{x}{30} \text{ hours},$$

and the time he spends driving home is

$$\text{Time to home} = \frac{\text{Distance to home}}{\text{Rate to home}} = \frac{x \text{ miles}}{45 \text{ mph}} = \frac{x}{45} \text{ hours}.$$

Therefore, the total time he spends driving is $\frac{x}{30} + \frac{x}{45}$ hours. Since he covers a distance of $2x$ over this time, his average rate is

$$\text{Average rate} = \frac{\text{Total distance}}{\text{Total time}} = \frac{2x \text{ miles}}{\frac{x}{30} + \frac{x}{45} \text{ hours}} = \frac{2x \text{ miles}}{\frac{5x}{90} \text{ hours}} = 36 \text{ miles per hour}.$$

This differs from our incorrect Bogus Solution because the Bogus Solution assumes Jack spends the same amount of *time* driving 30 mph and 45 mph. Instead, Jack covers the same *distance* driving 30 mph as he does when driving 45 mph. Since he covers this distance faster at the higher speed, he spends less time driving at that speed. Therefore, his average speed will be closer to his lower speed than to his higher speed. □

Sidenote: The **arithmetic mean** of two numbers is the average of the numbers, which equals the sum of the numbers divided by 2. The **harmonic mean** of two numbers is the reciprocal of the average of the reciprocals of the numbers. Our exploration of Problem 7.11 gives us an intuitive explanation for why the arithmetic mean of two positive numbers is always greater than or equal to the harmonic mean of those numbers. See if you can figure out why this must always be the case!

"Rate times time equals distance" isn't only applicable to problems involving motion.

Problem 7.12: Pippin and Sam are painting a fence. Sam could paint the whole fence alone in 12 hours. Pippin could paint the whole fence alone in 8 hours. Sam starts painting at 1 p.m. and Pippin joins him at 3 p.m. At what time do they finish?

Solution for Problem 7.12: First we figure out how much of the fence Sam has finished by the time Pippin starts helping. Since Sam can paint the whole fence in 12 hours, and paints alone for 2 hours before Pippin comes along, Sam has finished $2/12 = 1/6$ of the fence before Pippin starts helping. Therefore, they only have $5/6$ of the fence left.

Once Pippin joins in, Sam still paints $1/12$ of the fence per hour, but now Pippin also paints $1/8$ of the fence each hour (since Pippin alone can paint the whole fence in 8 hours). So, together, they paint

$$\frac{1}{12} + \frac{1}{8} = \frac{5}{24}$$

of the fence each hour. They must paint $5/6$ of the fence to finish, so the time, t, that they spend painting must satisfy

$$\left(\frac{5}{24}\right)t = \frac{5}{6}.$$

Solving, we find $t = 4$, so they will finish four hours after 3 p.m., at 7 p.m. □

What did that solution have to do with $rt = d$? First, we tackled the problem by looking at the amount of the job each person does in one hour. Sam does $1/12$ of the job in an hour and Pippin does $1/8$. These $1/12$ and $1/8$ are *rates*! They measure amount of work per hour, just like "miles per hour" measures distance traveled per hour.

Concept: Many problems involving work can be solved by considering the amount of work each worker does per some unit of time.

Once we know Sam's and Pippin's rates of work, we simply multiply by the time they work to find the total amount of work done. In other words,

$$\text{(Rate of work)} \times \text{(Time worked)} = \text{Amount of work done}.$$

This is exactly the same concept that we used to solve the first two problems in this section.

Problem 7.13: Flo and Carl each must read a 500-page book. Flo reads one page every minute. Carl reads one page every 50 seconds. Flo starts reading at 1:00, and Carl starts reading at 1:30. When will Carl catch up to Flo?

Solution for Problem 7.13: Solution 1: Examine each separately. When Carl starts reading, Flo has already read for 30 minutes; so, she has read 30 pages already. Let m be the number minutes after 1:30 that the two have been reading. Since Flo reads a page every minute and she has read 30 pages before 1:30, the number of pages she has read m minutes after 1:30 is $m + 30$. Carl reads a page every 50 seconds, so he reads one page every 5/6 of a minute. Since Carl reads a page every 5/6 of a minute, his reading rate is

$$\text{Carl's rate} = \frac{1 \text{ page}}{\frac{5}{6} \text{ minute}} = \frac{6}{5} \text{ pages per minute.}$$

Therefore, after m minutes, Carl has read $6m/5$ pages. If the two have read the same number of pages m minutes after 1:30, then we must have

$$m + 30 = \frac{6m}{5}.$$

Solving, we find $m = 150$, so Carl catches her 150 minutes after he starts, at 4:00.

Solution 2: Consider their relative rates. Since Carl reads 6/5 of a page each minute and Flo reads 1 page a minute, Carl catches up to her by $6/5 - 1 = 1/5$ page every minute. Flo starts with a 30 page head-start and Carl gains on her by 1/5 page per minute, so he'll catch her in

$$\frac{30 \text{ pages}}{\frac{1}{5}\,\frac{\text{pages}}{\text{min}}} = 150 \text{ minutes}$$

after he starts reading. Therefore, Carl will catch up to her at 4:00. □

Concept: If two objects are moving in a problem, sometimes it's easier to consider how the objects are moving relative to each other than to consider the two separately.

Specifically, in Problem 7.13, Flo and Carl are both "moving" through the book. Our second solution shows we can solve the problem quickly by considering how fast Carl is gaining on Flo.

Sometimes the people in the problem aren't the only things that are moving.

Problem 7.14: Barry owns a house on a river. The river flows at 3 miles per hour. Barry decides to start rowing downstream, with the current, at noon. He wants to return to his house at 5 p.m. At what time should Barry turn around and row home if he normally rows 5 miles per hour in water that has no current?

Solution for Problem 7.14: Although Barry rows 5 mph in water with no current, the moving water in the river will make him faster going downstream and slower going upstream. Specifically, he'll move 8 miles per hour downstream and 2 mph on his return trip upstream. We let d be the amount of time he rows downstream and u be the amount of time upstream. He rows for 5 hours, so $d + u = 5$. He also covers the same distance upstream as down, so $8d = 2u$. Therefore, we have the system of equations

$$\begin{aligned} d + u &= 5, \\ 8d &= 2u. \end{aligned}$$

Solving these equations gives $d = 1$ and $u = 4$, so he should turn around at 1 p.m., after he has rowed with the river for an hour. □

WARNING!! If objects (like Barry) are moving in a medium (like the river) that is also moving, we must take into account how the medium moves when determining the rate the object moves.

We end this section with a couple of classic rate problems.

Problem 7.15: Two trains, each moving at 20 miles per hour towards each other, are initially 60 miles apart. A bee starts at the front of one train, flies to the other train, then back to the first train, and so on. If the bee always flies at 30 miles per hour, how far does the bee fly before the trains collide?

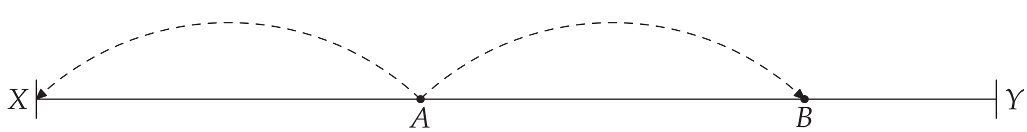
Solution for Problem 7.15: We could start by first finding how far the bee flies before reaching the second train, then computing how far it flies before it returns to the first train, then how far it flies going back to the second train, and so on. This looks like a pretty tough approach to take, so let's see if we can find another approach.

Since we need to find the distance the bee flies and we already know how fast the bee flies, we can solve the problem by figuring out how long the bee flies. Fortunately, that's easy! Since the two trains each move 20 mph, they approach each other at 40 miles per hour. Therefore, they will cover the 60 miles between them and collide in $60/40 = 1.5$ hours. Now we can compute the distance covered by the bee! The bee flies 30 miles per hour for 1.5 hours, so it flies $30(1.5) = 45$ miles before the trains collide. □

Concept: Rate times time equals distance. If you can find two of these three quantities, you have the other one. Therefore, if you're asked for one of these quantities but aren't sure how to find it, think about whether or not you can find the other two.

Problem 7.16: A man is running through a train tunnel. When he is $\frac{2}{5}$ of the way through, he hears a train that is approaching the tunnel from behind him at a speed of 60 mph. Whether he runs ahead or runs back, he will reach an end of the tunnel at the same time the train reaches that end. At what rate, in mph, is he running? *(Source: MATHCOUNTS)*

Solution for Problem 7.16: We don't know how long the tunnel is. We don't know how far the train is from entering the tunnel. It seems like we can't possibly have enough information to solve the problem. We could define variables for the man's speed, the length of the tunnel, and how far away the train is, then set up some equations and try to solve them. However, because we don't seem to have much information to begin with, we first try to get a better understanding of the problem by drawing a picture.



In the picture above, point A is where the man first hears the train, X is the end of the tunnel closest to the train, and Y the end of the tunnel farthest from the train. If the man runs to X, he'll get to X just as the train reaches X. If he instead runs the other way, he will get to point B. The distance from A to B

must equal that from A to X. Since A is $2/5$ of the way from X to Y, point B is 2 times this distance, or $4/5$ of the way from X to Y.

We know the man will reach Y at the same time as the train does, so he covers the remaining $1/5$ of the tunnel from B to Y in the same time the train covers the whole tunnel. Since the man must cover $1/5$ the distance that the train covers in the same amount of time, the man must move at $1/5$ the rate of the train, or $60/5 = 12$ mph. □

Concept: A good diagram can be an excellent problem solving tool.

Exercises

7.4.1 Jack drives at 40 mph for an hour, then at 50 mph for an hour. What is his average speed?

7.4.2 Alone, Brenda can dig a ditch in 5 hours. If Jack helps her, the two of them can dig the ditch in 3 hours. How long would it take Jack to dig the ditch himself?

7.4.3 Mr. Earl E. Bird leaves his house for work at exactly 8:00 a.m. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early.

(a) Suppose his house is x miles from work. Find, in terms of x, how long in minutes it takes Earl to get to work when he drives 40 miles per hour. What if he drives 60 miles per hour?

(b) Find x using the information you found in part (a).

(c) At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time? *(Source: AMC 10)*

7.4.4 A plane is traveling between City A and City B, which are 2000 miles apart. City A is due north of City B, and there is a strong, constant wind blowing due south. At constant speed, it takes a plane 5 hours to go from A to B with the wind at its tail, but 8 hours to go back when facing this headwind. What is the speed of the wind?

7.4.5 Bart is writing lines on a chalkboard that is initially empty. It ordinarily takes Bart 50 minutes to cover the whole board; however, today, Nelson is erasing the board while Bart is writing. Nelson can erase the board in 80 minutes by himself. If they work simultaneously, how long will it be until the whole board is covered?

7.4.6 When Kelsey is not on the moving sidewalk, she can walk the length of the sidewalk in 3 minutes. If she stands on the sidewalk as it moves, she can travel the length in 2 minutes. If Kelsey walks on the sidewalk as it moves, how many minutes will it take her to travel the same distance? Assume she always walks at the same speed. *(Source: MATHCOUNTS)*

7.4.7★ Sunny runs at a steady rate, and Moonbeam runs m times as fast, where m is a number greater than 1. If Moonbeam gives Sunny a head start of h meters, how many meters must Moonbeam run to overtake Sunny? (Give your answer as an expression in terms of h and m.) *(Source: AHSME)* **Hints:** 68