

CHAPTER 10. GEOMETRIC PROBABILITY

Exercises

10.2.1 Let \overline{AB} be a line segment of length 10. A point P is chosen at random on \overline{AB} . What is the probability that P is closer to the midpoint of \overline{AB} than to either endpoint?

10.2.2 A real number x is chosen at random such that $-2 \leq x \leq 5$. What is the probability that $x^2 < 2$?

10.2.3 A real number y is chosen at random such that $0 \leq y \leq 100$. What is the probability that $y - \lfloor y \rfloor \geq \frac{1}{3}$? **Hints:** 63

10.2.4★ Let \overline{CD} be a line segment of length 6. A point P is chosen at random on \overline{CD} . What is the probability that the distance from P to C is smaller than the *square* of the distance from P to D ? **Hints:** 130

10.2.5★ Let P be a point chosen at random on the line segment between the points $(0, 1)$ and $(3, 4)$ on the coordinate plane. What is the probability that the area of the triangle with vertices $(0, 0)$, $(3, 0)$ and P is greater than 2? **Hints:** 113

10.3 Probability Using Areas

Problems

Problem 10.4: Point C is chosen at random atop a 5 foot by 5 foot square table. A circular disk with a radius of 1 foot is placed on the table with its center directly on point C . What is the probability that the entire disk is on top of the table (i.e. that none of the disk hangs over an edge of the table)?

Problem 10.5: Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{3}{2}$?

Problem 10.6: My friend and I are hoping to meet for lunch. We will each arrive at our favorite restaurant at a random time between noon and 1 p.m., stay for 15 minutes, then leave. We want to determine the probability that we will meet each other while at the restaurant. (For example, if I show up at 12:10 and my friend shows up at 12:15, then we'll meet; on the other hand, if I show up at 12:50 and my friend shows up at 12:20, then we'll miss each other.)

- How can we represent the space of possible outcomes as a geometric region?
- What is the portion of the region from (a) which corresponds to me and my friend meeting?
- What are the areas of the regions from (a) and (b)?
- What is the probability that we meet?

We can calculate probabilities using areas in pretty much the same way that we calculate probabilities using lengths. As with lengths, we will have a region corresponding to "total outcomes." Within this

region, we will have a smaller region corresponding to "successful outcomes." Then

$$P(\text{success}) = \frac{P(\text{Area of successful outcomes region})}{P(\text{Area of total outcomes region})}.$$

We can model a great many more problems using areas. Let's look at a few examples.

Problem 10.4: Point C is chosen at random atop a 5 foot by 5 foot square table. A circular disk with a radius of 1 foot is placed on the table with its center directly on point C . What is the probability that the entire disk is on top of the table (i.e. that none of the disk hangs over an edge of the table)?

Solution for Problem 10.4: The "total outcomes" region is easy: it's just the surface of the table, so it's a square region with side length 5.

The "successful outcomes" region is trickier. We can draw a diagram as in Figure 10.4.

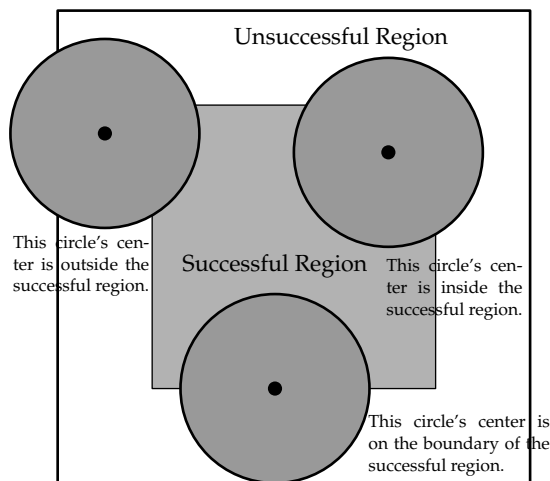


Figure 10.4: Table and some disks for Problem 10.4

We can see that the disk will be entirely on the table if and only if C is at least 1 foot away from each edge of the table. Therefore, C must be within a central square region of side length 3, as shown in Figure 10.4.

So now we can compute the probability:

$$P(\text{success}) = \frac{P(\text{Area of successful outcomes region})}{P(\text{Area of total outcomes region})} = \frac{3 \times 3}{5 \times 5} = \frac{9}{25}.$$

□

Problem 10.4 is an example of a problem which is stated in terms of geometry. But we can also use areas to solve algebraic probability problems similar to Problem 10.2.

Problem 10.5: Suppose two numbers x and y are each chosen such that $0 < x < 1$ and $0 < y < 1$. What is the probability that $x + y > \frac{3}{2}$?

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Solution for Problem 10.5: Since there's nothing discrete in this problem that can be counted, we'll need to use geometry.

The "total outcomes" region is the region in the xy -plane with $0 < x < 1$ and $0 < y < 1$. This is the interior of the square with corners $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$, as shown in Figure 10.5.

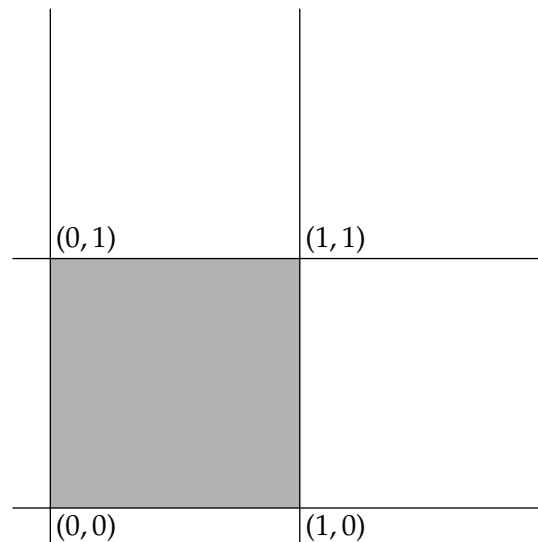


Figure 10.5: Total outcomes region for Problem 10.5

Now we need to describe the region corresponding to successful outcomes. We need $x + y > \frac{3}{2}$, so this will be the region inside the square of Figure 10.5 which is above the line $x + y = \frac{3}{2}$. We will add this to our diagram in Figure 10.6, and see that our successful region is the interior of a triangle.

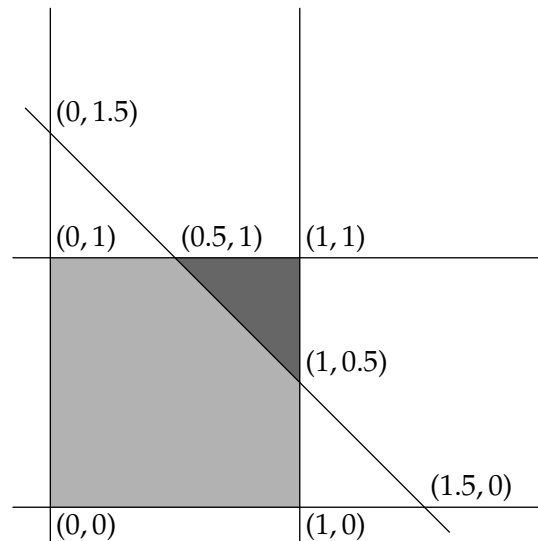


Figure 10.6: Total and successful outcomes regions for Problem 10.5

We can now calculate the areas. The area of the square is 1. The area of the triangle is $\frac{1}{8}$. Therefore

the probability is

$$P(\text{success}) = \frac{P(\text{Area of successful outcomes region})}{P(\text{Area of total outcomes region})} = \frac{\frac{1}{8}}{1} = \frac{1}{8}.$$

□

The real power of geometric methods is taking word problems and recasting them as geometry problems that we can work with. The next problem is a classic example of this.

Problem 10.6: My friend and I are hoping to meet for lunch. We will each arrive at our favorite restaurant at a random time between noon and 1 p.m., stay for 15 minutes, then leave. What is the probability that we will meet each other while at the restaurant? (For example, if I show up at 12:10 and my friend shows up at 12:15, then we'll meet; on the other hand, if I show up at 12:50 and my friend shows up at 12:20, then we'll miss each other.)

Solution for Problem 10.6: Once again, because we have infinitely many possibilities, we won't be able to use regular counting, so we'll need to look for a geometric approach.

We can make a graph plotting my arrival time and my friend's arrival time. We'll put my arrival time on the x -axis and my friend's arrival time on the y -axis. The result is Figure 10.7.

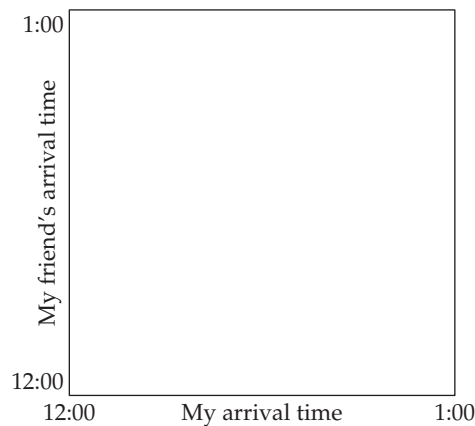


Figure 10.7: Graph for meeting times

We now need to figure out what portion of Figure 10.7 corresponds to me and my friend meeting. If my friend and I are to successfully meet, he must arrive no earlier than 15 minutes before I arrive, and no later than 15 minutes after I arrive. If unsure how to proceed at this point, we could experiment with a few example arrival times to see what the picture is going to look like.

Concept: If unsure how to proceed with a problem, try a few simple examples.



Here are a few examples:

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My arrival time	My friend's successful arrival times
12:00	12:00-12:15
12:10	12:00-12:25
12:20	12:05-12:35
12:30	12:15-12:45
12:40	12:25-12:55
12:50	12:35-1:00
1:00	12:45-1:00

We can plot these on our graph, and the result is shown in Figure 10.8. Based on what we've drawn in Figure 10.8, we have a pretty good picture of what the successful meeting region is going to look like. We can essentially "fill in" the picture, to get Figure 10.9.

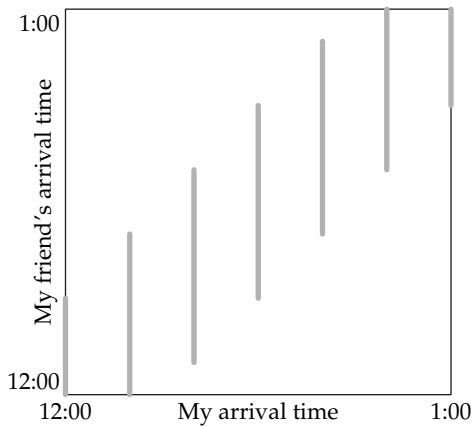


Figure 10.8: Some possible meeting times

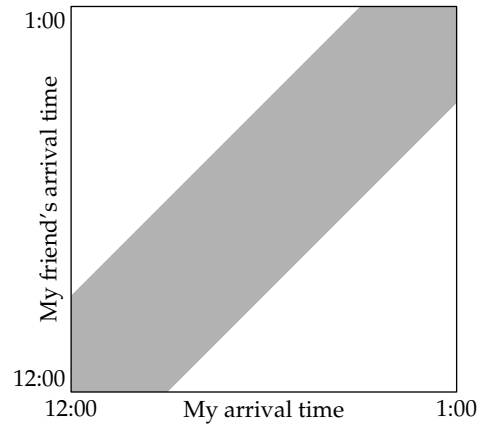


Figure 10.9: Successful meetings

To be precise, we can write the condition for a successful meeting as an equation:

$$y - 15 \leq x \leq y + 15,$$

where x is my arrival time and y is my friend's arrival time, expressed in minutes past 12:00. So the "successful outcomes" region is the region inside the square that's below the line $y = x + 15$ and above the line $y = x - 15$. Figure 10.10 shows the result after we add these lines to our diagram.

Extra! *As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection. – Joseph-Louis Lagrange*

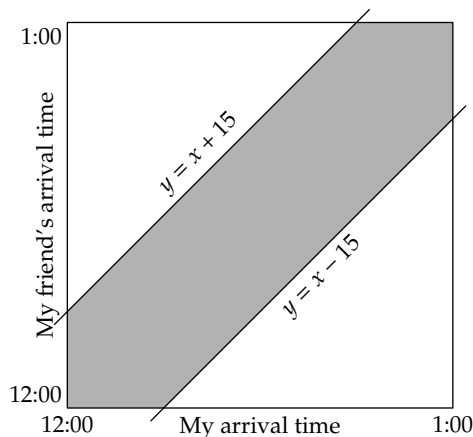


Figure 10.10: Successful meeting times

Figure 10.10 shows a two-dimensional representation of all possible times my friend and I can show up, along with a shaded band representing success. Notice, as expected, that the graph is symmetrical. Because there is nothing in the problem to indicate a difference between my situation and that of my friend, we should expect to see no difference if we were to swap axes.

We can now compute the areas of the two regions, and then divide to get the probability. The “total outcomes” region is just a square with side length 60, so its area is $60 \times 60 = 3600$. The “successful outcomes” region is a hexagon, for which it is somewhat unpleasant to calculate the area. But we can pretty easily calculate the area of the “unsuccessful outcomes” region, which is just two triangles. Each half of the region is a right isosceles triangle with side length 45, with area $\frac{1}{2} \times 45 \times 45$. So the “unsuccessful outcomes” region has total area $2 \times \frac{1}{2} \times 45 \times 45 = 2025$.

Therefore the “successful outcomes” region has area $3600 - 2025 = 1575$, and finally the probability of success is

$$P(\text{we meet for lunch}) = \frac{1575}{3600} = \frac{7}{16}.$$

□

Exercises

10.3.1 A point (x, y) is chosen at random inside the square with vertices $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$. What is the probability that

- $x + y \leq 0.5$?
- $x + 2y \geq 1$?
- $|x - y| \leq 0.2$?
- $x^2 + y^2 < 1$?
- the distance from (x, y) to the center $(0.5, 0.5)$ of the square is less than 0.5?
- the distance from (x, y) to $(0, 1)$ is greater than 1?

10.3.2 Steve’s kitchen floor has a tile pattern of square tiles of side length 10 cm. Steve drops a penny (which has radius 1 cm) on the floor. What is the probability that the penny lies entirely within one tile?