

Life of Fred
Advanced Algebra

Stanley F. Schmidt, Ph.D.



Polka Dot Publishing

What's in Advanced Algebra?

All kinds of stuff. It's the second half of algebra. You've seen the first half, and therefore, things like $2x = 14$ are not very scary anymore. In beginning algebra we've already done most of the classic word problems such as . . .

JENNIFER CAN DIG A DITCH IN 4 HOURS.
JASON CAN DIG IT IN 5 HOURS. IF THEY
WORK TOGETHER HOW LONG WILL IT TAKE?

or

JASON RUNS DOWN THE HALL AT 5 MPH.
WHEN HE'S 50 FEET AWAY, JENNIFER RUNS
AFTER HIM AT 6 MPH. HOW LONG BEFORE
THEY'RE HAPPY?

We've already learned 94.7% of factoring. The only thing left is the factoring of $x^3 + y^3$, which is $(x + y)(x^2 - xy + y^2)$. Oops! I guess you've just finished factoring.

You've gone through the agony of learning to add and multiply algebraic fractions:

$$\frac{x+2}{x+5} + \frac{x+1}{x+4} = \frac{(x+2)(x+4)}{(x+5)(x+4)} + \frac{(x+1)(x+5)}{(x+4)(x+5)} = \frac{(x+2)(x+4) + (x+1)(x+5)}{(x+5)(x+4)}$$

and the terror of the quadratic formula: $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

The further you go in math, the less memorizing and the less computational cookbook stuff you encounter. You will find that *understanding* rather than just being a good tape recorder starts to matter most.

In Fred's everyday life in this book, he runs into things that would baffle a beginning algebra student. For example, in chapter three you learn how to solve $2^x = 5$. In chapter six you learn how to battle the dreaded Snow King using a Waddle-Ray which can be obtained at your local doughnut store. In chapter nine we add up an infinite number of numbers, such as $1/2 + 1/6 + 1/18 + 1/54 + \dots$, and we get an answer! A finite answer. Not your usual old stuff.

A Note to Students

Hi! This is going to be fun.

Fred and you are both veterans. He has just received an honorable discharge from the army and is taking the bus home to KITTENS University in Kansas. You are a veteran of beginning algebra. You are about to join him on that bus ride.

On the two-day ride you will experience all of advanced algebra—everything you need to know for trigonometry, for calculus, and for the rest of your life. You will have it all.

The supplies you'll need for the trip:

1. pencil or pen
2. paper
3. one of those basic handheld calculators that has $+$, $-$, \times , \div , and $\sqrt{\quad}$ keys. Last time I checked, the fancier solar models (that don't require a battery) cost about \$5.

It would be nice, but it's not required, to have one of the fancier handheld calculators (often called a scientific calculator) that has the keys: \sin , \cos , \log , Σ , and y^x . You will be able to use it occasionally in this book, but owning one won't be mandatory until the next course which is trigonometry. And that will be the last calculator that you'll ever need to get. When you get to calculus, which is right after trigonometry, you won't need to get any more machines.

When I studied algebra, my teacher told the class that we could reasonably expect to spend thirty minutes per page to master the material in the old algebra book we used. With the book you are holding, you will need two reading speeds: thirty minutes per page when you're learning algebra and whatever speed feels good when you're enjoying the life adventures of Fred.

Throughout the book are sections called *Your Turn to Play*, which are opportunities for you to interact with the material. That's what the supplies listed on the previous page are used for. Just reading the problems and the solutions that are furnished doesn't work. Answers are supplied to half of the questions in the Cities section at the end of each chapter.

The general rule is easy: All of the *Your Turn to Play* and the half of the Cities questions that have the answers supplied are yours to tackle.

Our story begins at noon on Monday. Start on page one and things will explain themselves nicely.

After 10 chapters you will have mastered all of advanced algebra.

Just before the Index (starting on p. 308) the **A.R.T.** section begins. (**A.R.T.** = All Reorganized Together)

This section very briefly summarizes advanced algebra. If you have to review for a final exam or you want to quickly look up some topic eleven years after you've read this book, the **A.R.T.** section is the place to go.

Contents

Chapter 1	Ratio, Proportion, and Variation	16
	median average	
	cross multiplying	
	constant of proportionality	
Chapter 1½	Looking Back	38
	exponents	
	square roots	
	rationalizing the denominator	
Chapter 2	Radicals	43
	radical equations	
	extraneous answers	
	The History of Mathematics—a story	
	irrational numbers	
	imaginary numbers	
Chapter 2½	Looking Back	70
	Venn Diagrams	
	disjoint sets	
	union	
	intersection	
	significant digits	
	scientific notation	
Chapter 3	Logarithms	82
	exponential equations	
	the laws of logs	
	three definitions of logarithm	
Chapter 3½	Looking Back	110
	graphing by point-plotting	
	ordered pairs, abscissa, ordinate, origin, quadrants	
Chapter 4	Graphing	116
	slope	
	distance between points	
	slope-intercept form of the line	
	double-intercept form of the line	
	point-slope form of the line	
	two-point form of the line	
	slopes of perpendicular lines	

Chapter 4½	Looking Back	135
	factoring	
	common factors	
	easy trinomials (of the form $x^2 + bx + c$)	
	difference of squares	
	grouping	
	harder trinomials (of the form $ax^2 + bx + c$)	
	fractions	
	simplifying	
	adding, subtracting	
	multiplying, dividing	
	complex fractions	
	equations	
	linear	
	fractional	
	quadratic	
	by factoring	
	pure quadratics	
	the quadratic formula	
	radical equations	
Chapter 5	Systems of Equations	147
	solving by elimination	
	solving by substitution	
	solving by graphing	
	inconsistent and dependent systems	
	solving by Cramer's rule	
	expanding determinants by minors	
Chapter 6	Conics	172
	ellipse	
	major and minor axes	
	vertices and foci	
	reflective property	
	circle	
	parabola	
	hyperbola	
	graphing inequalities in two variables	
	conic sections not centered at the origin	
Chapter 7	Functions	201
	definition	
	domain, codomain, range, image	
	1-1, onto, 1-1 correspondence	
	inverse functions	
	relations	
	identity function	

Chapter 7½	Looking Back	237
	long division of polynomials	
Chapter 8	Linear Programming, Partial Fractions, and Math Induction	241
	the four cases for partial fractions	
	numerals vs. numbers	
	very large numbers	
Chapter 9	Sequences, Series, and Matrices	268
	arithmetic	
	last term formula	
	sum	
	matrix addition and multiplication	
	geometric	
	last term	
	sum of finite series	
	sum of infinite series	
	sigma notation	
Chapter 10	Permutations and Combinations	284
	the fundamental principle	
	factorial	
	$P(n, r)$	
	$C(n, r)$	
	permutations where some of the items are identical	
	binomial formula	
	Pascal's triangle	
Combinatorially Yours		307
	the hardest problem in advanced algebra	
A.R.T. (All Reorganized Together) section		308
	a quick summary of all of advanced algebra	
Index		317

Chapter One

Ratio, Proportion, and Variation

Fred looked out the bus window. The cold, white Texas landscape might have seemed bleak to many people, but to him it was a joy. He was heading north—back to his home in Kansas.

He thought about the last four days. Friday had been his sixth birthday. So much had happened since then: his “abduction” into the army, all the new friends he had met, his hurt rib, and his honorable discharge.*

Now he could look out at the passing telephone poles and just imagine them as a clock ticking away the six hours till he reached his office at KITTENS University (Kansas Institute for Teaching Technology, Engineering and Natural Sciences), where he has lived for the last five years.

It would be good to get out of his hospital nightshirt with the little blue and green frogs all over it. Tomorrow would be Tuesday and maybe by then his rib wouldn’t hurt so badly. With a good night’s sleep and a fresh bunch of clothes, he’d meet his 9 A.M. trigonometry class.

The telephone poles whooshed by, one after another. He looked out of the bus window and unconsciously began to count them: five poles passed for every three beats of his heart. (He could feel his pulse as little stings in his hurt rib.) The **ratio** of the passing poles to the heartbeats was 5:3. Ratio means division, so 5:3 could also be written as $5 \div 3$.

His eyes began to close, shutting out the snowy scene. A little nap would help pass the time. Five-thirds would become ten-sixths would become fifteen-ninths. He’d soon be asleep.

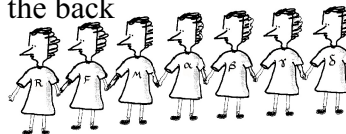
“Hey! How old are you?”

Fred was startled by the half-shouted question. He received a little poke in the ribs and then he was fully awake.

“I said how old are you,” the little girl repeated.

“I’m six. I just turned six last Friday.”

She said, “Oh” and ran to her friends in the back of the bus. They were all about four years old and were all dressed identically in gray-brown dresses. They giggled and chattered.



* This story is told in *Life of Fred: Beginning Algebra*

Fred might easily have been mistaken for a four-year-old. He had always been less than the **median** weight for his age. (The median weight means that half the people are heavier and half the people are lighter than that weight. The median is one of the three kinds of averages studied in beginning statistics.) Fred, at 37 pounds, was definitely less than the median weight for his age. Maybe only 4% of boys his age weighed less than he did.

He noticed that the ratio of the telephone poles that the bus was passing to his pulse was now 5:4. It wasn't that the bus was going faster, but simply that his heart was beating more quickly. Getting awakened with a question and a poke would cause most people's hearts to beat faster.

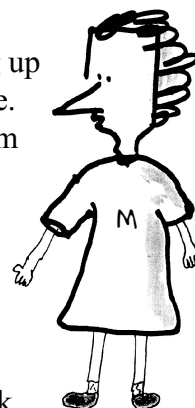
Oh well, he thought, and after a few moments he began to drift back to sleep.

He could hear her coming. Some little girl running up the aisle to his seat. It was a different girl than the first one. Instinctively, he put his arms around his ribs to protect them against further assault.

"What's your name?" she blurted. She had been sent on a mission to find this out.

Fred, who had read all the James Bond books, thought of answering, "Gauss, Fred Gauss," but instead he simply said, "Fred."

"Oh" was her only response, and she ran to the back of the bus to report her findings to her girlfriends.



Fred was so used to being around the students at KITTENS that these four-year-olds seemed to him to be so . . . he couldn't think of the word. They seemed to be so *immature*.

A woman, also wearing one of the gray-brown uniform dresses, came up to Fred and smiled. "Hi. My name is Cheryl Mittens. I hope my little girls haven't been bothering you."

"Not too much. Could you tell me what's going on? Are they just playing or something?"

"Well, you might call it that," Cheryl said. "They're working on earning a badge for their uniforms. It's the How-to-Find-a-Husband badge. The first requirement is to learn to make contact with some fellow that they want to get to know."

“Oh,” said Fred. (He was starting to sound like the girls.) “But I’m 50% older than these girls. (He had done the math in his head: $4 \times 1.50 = 6$.) Aren’t they supposed to pick a guy that’s close to themselves in age? I read in a marriage manual, Professor Eldwood’s *Treatise on Modern Courtship and Marriage*, 1846, that the prospective bride should be at least 90% of the age of the groom. That would make the ratio of their ages 9:10. Right now, the ratio of their ages to mine is 4:6.”

Cheryl laughed. “I guess you’re right. The girls are only playing at getting married right now. Don’t worry. They’ll never be 90% of your age.”

Fred’s heart raced at the thought of marriage to one of those *children*. Maybe he’d be lucky and he’d be a hundred years old before they were grown up enough to be of the appropriate age. Then he wondered how long it really would be before they were 90% of his age—before the ratio of their ages to his age was 9:10. He let x equal the number of years from now until that happened. In x years, the girls would be $4 + x$ years old and he would be $6 + x$ years old. In x years, the ratio $4 + x : 6 + x$ would be the same as 9:10.

$$\frac{4 + x}{6 + x} = \frac{9}{10}$$

Two ratios set equal to each other is called a **proportion**. When we solved fractional equations in beginning algebra, we found an expression that all the denominators would evenly divide into and multiplied every term by that expression. In this case $10(6 + x)$ will do the trick:

$$\frac{(4 + x)10(6 + x)}{6 + x} = \frac{9 \cdot 10(6 + x)}{10}$$

Then all the denominators disappear

$$(4 + x)10 = 9(6 + x)$$

Multiply it out using the distributive property

$$40 + 10x = 54 + 9x$$

the distributive property

$$a(b + c) = ab + ac$$

Subtract $9x$ from each side

$$40 + x = 54$$

Subtract 40 from each side

$$x = 14$$

A.R.T.

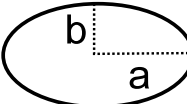
All Reorganized Together

A Super-condensed and Reorganized-by-Topic Overview of Advanced Algebra (Highly abbreviated)

Topics:

- C**onics
- F**actoring
- F**ractions
- F**unctions
- G**raphing
- I**maginary Numbers
- L**inear Programming
- L**ogs
- M**ath Induction
- M**atrices
- P**artial Fractions
- P**ascal's Triangle
- P**ermutations & Combinations
- R**adical Equations
- R**atios & Proportions
- S**equences and Series
- S**ignificant Digits
- S**ystems of Equations

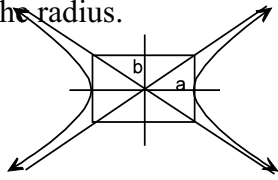
Conics

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 

The larger of a or b is called the semi-major axis. The other one is called the semi-minor axis. (p. 173 and p. 195)

Circle $x^2 + y^2 = r^2$ where r is the radius. (p. 178)

Parabola $y = ax^2$   (p. 186)

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  (p. 189)

Factoring

$$(a + b)(c + d) = ac + ad + bc + bd \quad (\text{p. 135})$$

memory aid: boys and girls dating

$$\text{Common factor: } 4x^2y - 6xy = 2xy(x - 3) \quad (\text{p. 136})$$

$$\text{Easy trinomials: } x^2 + 5x + 6 = (x + 2)(x + 3) \quad (\text{p. 136})$$

memory aid: 2 and 3 add to 5 and multiply to 6

$$\text{Difference of squares: } 49x^2 - 25y^2 = (7x + 5y)(7x - 5y) \quad (\text{p. 137})$$

$$\text{Grouping: } 6x^3 - 15x^2 + 4x - 10 \quad (\text{p. 138})$$

$$= 3x^2(2x - 5) + 2(2x - 5)$$

$$= (2x - 5)(3x^2 + 2)$$

$$\text{Harder trinomials: } 6x^2 + 29x + 35 \quad (\text{p. 138})$$

Split the $29x$ into two numbers that add to $29x$ and multiply to $(6x^2)(35)$

$$6x^2 + 15x + 14x + 35$$

and finish the factoring by grouping

$$3x(2x + 5) + 7(2x + 5)$$

$$(2x + 5)(3x + 7)$$

Fractions

Adding fractions, see p. 141.

Subtracting fractions, see p. 142.

Multiplying fractions, see p. 142.

Dividing fractions, see p. 143.

Solving fractional equations:

$$\frac{1}{12} + \frac{1}{16} + \frac{1}{24} = \frac{1}{x} \quad (\text{p. 144})$$

Multiplying by $48x$ to clear

$$\text{all the denominators } \frac{1 \cdot 48x}{12} + \frac{1 \cdot 48x}{16} + \frac{1 \cdot 48x}{24} = \frac{1 \cdot 48x}{x}$$

$$\text{which simplifies to } 4x + 3x + 2x = 48$$

If you multiply an equation by an expression containing variables, you must check your answers in the original problem to eliminate extraneous roots.

Functions

A function is any rule which assigns to each element of the domain exactly one element of the codomain. (p. 204)

1-1 functions (one-to-one functions) are those functions where no two elements in the domain have the same image. (p. 212)

The range of a function is the set of all the images. It is a subset of the codomain of the function. (p. 227, problem 5)

If f is a 1-1 function, we may create an inverse function f^{-1} .

The inverse function maps the elements in the range of f back to the domain of f so that $f^{-1}(f(x)) = x$ for all x in the domain. (p. 212)

A function is onto the codomain if every element of the codomain is the image of at least one element in the domain. (p. 224)

Functions that are both 1-1 and onto are called 1-1 correspondences. The domain and codomain of such functions must have the same number of elements. (p. 228)

A function may also be defined as a set of ordered pairs in which no two ordered pairs have the same first coordinate. If the ordered pairs of a function are graphed, no two points on the graph will be directly over each other. Any set of ordered pairs is called a relation. (pp. 230–232)

An identity function is a function where the domain equals the codomain and each element of the domain is mapped to itself. $f(x) = x$ (p. 232, problem 3)

$f:A \rightarrow B$ means the name of the function is f , the domain is A and the codomain is B . (p. 210)

Graphing

first coordinate = x coordinate = abscissa (p. 115)

second coordinate = y coordinate = ordinate

$(0, 0)$ = origin

The four quadrants:

II	I
III	IV

Slope = $\frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$ between the points (x_1, y_1) and (x_2, y_2) .

Distance between the points (x_1, y_1) and $(x_2, y_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Slope-intercept form of the line $y = mx + b$ (p. 123)

Double-intercept form $\frac{x}{a} + \frac{y}{b} = 1$ (p. 123)

Point-slope form $m = \frac{y - y_1}{x - x_1}$ (p. 126)

Two-point form of the line $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ (p. 128)

Two lines are perpendicular iff their slopes are negative reciprocals of each other. (p. 131)

Inequalities (p. 192)

Index

abscissa	115
absolute value	113, 213
approximately equal \approx	51
Argand diagram	62, 70
arithmetic series (progression)	
a, l, d, n, s	270
finding l	271
finding s	271
big numbers	257
binomial formula	300
Cantor, Georg	63
circle	178
cleaning your computer	22
combination	292
formula	293
common factor	136
completing the square	194
complex number plane	62
complex numbers	62, 74
conic sections	196
constant of proportionality	22
constant of variation	22
constellations (the 88 official ones)	
.	125
cross multiplying	19
degree of a polynomial	237
dependent variable	127
determinants	
evaluating 2×2	159
evaluating 3×3	160
expanding by minors	161
difference of squares factoring	137
distance between two points	122
double-intercept form of the line	123
ellipse	172
foci	181
major axis	198
reflective property	184
semi-major axis	173
semi-minor axis	173
standard form	195
vertex	173
equations	
fractional	144
general quadratic	145
linear	144
pure quadratic	144
quadratic by factoring	144
quadratic formula	145
radical equations	145
radical equations with two radicals	
.	146
exponential equation	84
exponents	39, 40
extraneous answers	46, 47
factorial	287
0!	287
gamma function	287
factoring	
common factor	136
difference of squares	137
easy trinomials	136
grouping	138
harder trinomials	138
multiplying binomials	135
fifty-foot tall Rita	30-33
fractions	
adding	141
complex fractions	143
dividing	143
multiplying	142
partial fractions	243
simplifying	140
subtracting	142
function	
1-1	212
1-1 correspondence	228
as a graph	232
as ordered pairs	230
codomain	204
definition	204
domain	204
f(x) notation	211
identity function	232
image	205
inverse	212

Index

onto	224	product rule	92
pre-image	227	quotient rule	93
range	227	second definition	102
relation	231	third definition	105
fundamental principle	285	long division of polynomials	237-240
geometric progression	277	MacDonald, Jeanette	236
finding s	278, 279	math induction	253-256
finding sum of an infinite geometric progression	281	matrix	273
geometric sequence	277	adding	274
finding the last term	277	dimension	281
Gormenghast Trilogy	266	entry	274
graphing		multiplying	275
abscissa	115	rows and columns	274
inequalities	192, 193	median average	17, 34
ordered pair	114	metonymy	83
ordered triple	114	natural numbers	53, 74
ordinate	115	Newton's first law	29
origin	115	numbers vs. numerals	249
plane in three dimensions	155	Omar Khayyam	176
quadrants	115	onomatopoeia	284
grouping factoring	138	ordered pair	114
harmonic progression	282	ordinate	115
hyperbola	189	origin	115
asymptotes	189	parabola	112, 167, 186
standard form	197	vertex	187
imaginary numbers	61	Pascal, Blaise	302
complex numbers	62	Pascal's triangle	
pure imaginary	62, 74	binomial formula	298
independent variable	127	chart	300
integers	54, 74	coin tosses	299
irony	84	entries are $C(n, r)$	297
irrational numbers	60	perimeter of a triangle	46
Kilmer, Joyce	128	permutation	289
lateral surface area of a cone	50	formula	290
linear programming		with some items alike	293, 294
constraints	258	point-plotting	110
objective function	265	point-slope form of the line	126
the second step after graphing the constraints	261	proportion	18
lines--all the formulas	131	cross multiplying	19
logs		pure quadratic	50
antilog	97	Pythagoras	57
change-of-base rule	104	Pythagorean theorem	42
common logs	101	quadrants	115
first definition	101	quadratic equations	
power rule (birdie rule)	86	by factoring	144
		pure	144
		quadratic formula	145

Index

radical equation	45
how to solve	47
ratio	16
rational numbers	55, 74
real numbers	60, 74
reciprocal	282
relation	231
Russell, Bertrand	224
scientific notation	78
sector	216
sigma notation	279
dummy variable	282
significant digits	75-79
in computing	79
slope	117
of perpendicular lines	130, 131
slope-intercept form of a line	123
square roots	41
rationalizing the denominator	41
subscripts	120
systems of equations	
Cramer's rule	156
dependent	151
determinant	156
inconsistent	151
solved by elimination	149
solved by graphing	149
solved by substitution	149
tachyon	63
tax on sleeping and eating	170
transfinite numbers	229
two-point form of the line	128
unit analysis	64
variation	
constant of proportionality	22
constant of variation	22
Venn Diagrams	70-74, 296
intersection	71
subset	72
union	71
vertex of a parabola	187
whole numbers	74