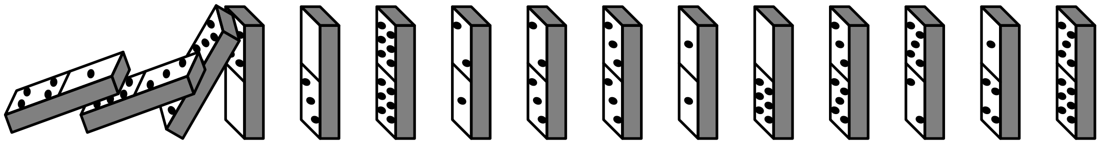

It's not the voting that's democracy; it's the counting – Tom Stoppard

CHAPTER 3 Correcting for Overcounting

3.1 Introduction

Often in counting problems, the most convenient solution is to count too much, and then somehow correct for the overcounting. We've already used this approach with complementary counting, in which we use subtraction to remove the items which we don't want.

However, there are many other counting methods which involve deliberately overcounting and then correcting for our overcounting. Unlike complementary counting, which uses subtraction, the methods that we'll discuss in this chapter use division to correct for overcounting.

We'll also talk about using overcounting to count how many ways we can place objects in a circle, or on a keychain, or how to count ways to place objects in a row when some of the objects are identical.

3.2 Permutations with Repeated Elements

Problems

Problem 3.1: How many possible distinct arrangements are there of the letters in the word DOG?

Problem 3.2: How many possible distinct arrangements are there of the letters in the word BALL?

Problem 3.3: How many distinct arrangements are there of TATTER?

Problem 3.4: How many distinct arrangements are there of PAPA?

Let's start with a problem that we should (hopefully) already know how to do.

Problem 3.1: How many possible distinct arrangements are there of the letters in the word DOG?

Solution for Problem 3.1: We could list them: DOG, DGO, ODG, OGD, GDO, GOD. Or we could have noticed that we have 3 ways to pick the first letter, 2 ways left to pick the second, and 1 way to pick the third, for a total of $3! = 6$ ways. □

Problem 3.1 is just a basic permutation problem. It is essentially the same as Problem 1.12 from Chapter 1. Let's add a new wrinkle to this type of problem.

Problem 3.2: How many possible distinct arrangements are there of the letters in the word BALL?

Solution for Problem 3.2: First, let's see why the answer is not simply 4!. What's wrong with the following solution?

Bogus Solution: Let's proceed as we did in Problem 3.1. We have 4 ways to pick the first letter, 3 ways to pick the second, and so on, for a total of 4! possibilities.

Unfortunately, this method overcounts. The reason for this is that two of our letters are the same.

Let's pretend that the two L's are different, and call them L_1 and L_2. So our word BALL is now really BAL_1L_2. In making the expected 4! arrangements, we make both BAL_1L_2 and BAL_2L_1. But when we remove the numbers, we have BALL and BALL, which are the same.

So we've overcounted and we need to correct for this. How?

Every possible arrangement of BALL is counted twice among our arrangements of BAL_1L_2. For example, LLAB is counted as both L_1L_2AB and L_2L_1AB, LABL is counted as both L_1ABL_2 and L_2ABL_1, and so on for every possible arrangement of BALL. We can see this in Figure 3.1:

BAL_1L_2, BAL_2L_1	$\Rightarrow$ BALL	BL_2AL_1, BL_1AL_2	$\Rightarrow$ BLAL
BL_1L_2A, BL_2L_1A	$\Rightarrow$ BLLA	ABL_1L_2, ABL_2L_1	$\Rightarrow$ ABLL
L_1BAL_2, L_2BAL_1	$\Rightarrow$ LBAL	L_1BL_2A, L_2BL_1A	$\Rightarrow$ LBLA
AL_1BL_2, AL_2BL_1	$\Rightarrow$ ALBL	L_1ABL_2, L_2ABL_1	$\Rightarrow$ LABL
L_1L_2BA, L_2L_1BA	$\Rightarrow$ LLBA	AL_1L_2B, AL_2L_1B	$\Rightarrow$ ALLB
L_1AL_2B, L_2AL_1B	$\Rightarrow$ LALB	L_1L_2AB, L_2L_1AB	$\Rightarrow$ LLAB

Figure 3.1: BALL's with different L's

Thus, the number of arrangements of BAL_1L_2 is exactly twice the number of arrangements of BALL. So to get the number of arrangements of BALL, we have to divide the number of arrangements of BAL_1L_2 by 2.

So the number of arrangements of BALL is $4!/2 = 12$. □

Now we have another counting strategy to go with counting the items we don't want – strategically counting multiples of the items we do want, then dividing. Because the L's can be arranged in 2 different ways, we need to divide by 2 to correct for the overcounting.

As we've said many times, don't blindly memorize this technique! Take the time to thoroughly understand why it works. Then it will become second nature.

Here's another example, where this time a letter occurs 3 times.

Problem 3.3: How many distinct arrangements are there of TATTER?

Solution for Problem 3.3: If we pretend that all the T's are different (say, T_1, T_2, and T_3), then there are 6! possible arrangements of $T_1AT_2T_3ER$.

However, for any given arrangement, the 3 T's can be rearranged in 3! ways, meaning that every arrangement of TATTER corresponds to 3! arrangements of $T_1AT_2T_3ER$. An example is in Figure 3.2.

$T_1AT_2T_3ER$	$T_2AT_1T_3ER$	$T_3AT_1T_2ER$	all correspond to TATTER
$T_1AT_3T_2ER$	$T_2AT_3T_1ER$	$T_3AT_2T_1ER$	

Figure 3.2: TATTER's with different T's

Thus, our 6! ways to arrange $T_1AT_2T_3ER$ counts each arrangement of TATTER 3! times. Hence, we divide 6! by 3! so that we'll only count each arrangement of TATTER once. Therefore, the number of arrangements of TATTER is $6!/3! = 120$. □

Problems 3.2 and 3.3 are examples of **strategic overcounting**. We notice that we can count each item a certain number of times in an easy way, then we divide by that number. For example, in counting the arrangements of TATTER, we know that 6! counts each arrangement exactly 3! ways, so our desired number of distinct arrangements is $6!/3!$. We'll be seeing this idea again. And again. And again.

To throw one more twist into these types of countings, let's see what happens if more than one item is repeated.

Problem 3.4: How many distinct arrangements are there of PAPA?

Solution for Problem 3.4: Again, we pretend that the letters are all different, and we have $P_1A_1P_2A_2$. We have 4! permutations (since all the letters are different).

But how many arrangements of $P_1A_1P_2A_2$ (where the P's and A's are considered different) correspond to a single arrangement of PAPA (where the P's and A's are identical)?

PAPA is counted in 4 different ways: as $P_1A_1P_2A_2$, $P_1A_2P_2A_1$, $P_2A_1P_1A_2$, and $P_2A_2P_1A_1$.

Rather than list out the possibilities, we could have reasoned as follows: For the 2 P's, each possibility is counted $2! = 2$ times, and for the 2 A's, each of these 2 possibilities is counted $2! = 2$ times, for a total of $2 \times 2 = 4$ ways. (Make sure you see why it isn't "$2 + 2 = 4$" ways.)

Therefore, there are 4! ways to arrange the 4 letters $P_1A_1P_2A_2$; this counts each arrangement of PAPA

$2! \times 2! = 4$ times, so we have $4!/(2! \times 2!) = 6$ ways to arrange PAPA. □

Exercises

3.2.1

(a) List all of the arrangements of the letters in the word EDGE.

(b) Compute the number of arrangements of the letters in the word E_1DGE_2, in which the two E's are considered different.

(c) How are your answers related?

3.2.2

(a) How many ways are there to arrange the letters of the word $BA_1N_1A_2N_2A_3$, in which the three A's and the two N's are considered different?

(b) How many ways are there to arrange the letters in the word BANANA, in which the three A's and the two N's are considered identical?

3.2.3 For each of the following words, determine the number of ways to arrange the letters of the word.

(a) COT

(b) WAR

(c) PIP

(d) THAT

(e) LULL

(f) CEASE

(g) TOOT

(h) TEPEE

(i) MADAM

(j) TARTAR

(k) ALABAMA

(l) MISSISSIPPI

3.2.4 I have 5 books, two of which are identical copies of the same math book (and all of the rest of the books are different). In how many ways can I arrange them on the shelf? **Hints:** 202

3.2.5 There are 8 pens along a wall in the pound. The pound has to allocate 4 pens to dogs, 3 to cats, and one to roosters. In how many ways can the pound make the allocation?

Extra! *Well, some mathematics problems look simple, and you try them for a year or so, and then you try them for a hundred years, and it turns out that they're extremely hard to solve.*
– Andrew Wiles